

Gabriel Ruiz Múgica

Structures I

Structural frame's analysis

Lecture notes

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Universidad
del País Vasco

Euskal Herriko
Unibertsitatea

Lecture notes

STRUCTURES I

STRUCTURAL FRAME'S ANALISYS

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1. INTRODUCTION

These notes have been developed throughout my teaching since 2009. This is a collection, revised and corrected, of my theoretical classes and the practical exercises that I have put together in this period.

The content of the subject is an introduction to the analysis and resolution of building frame structures. The teaching approach used in these notes is simple: structures calculation's theory must be learned by practice.

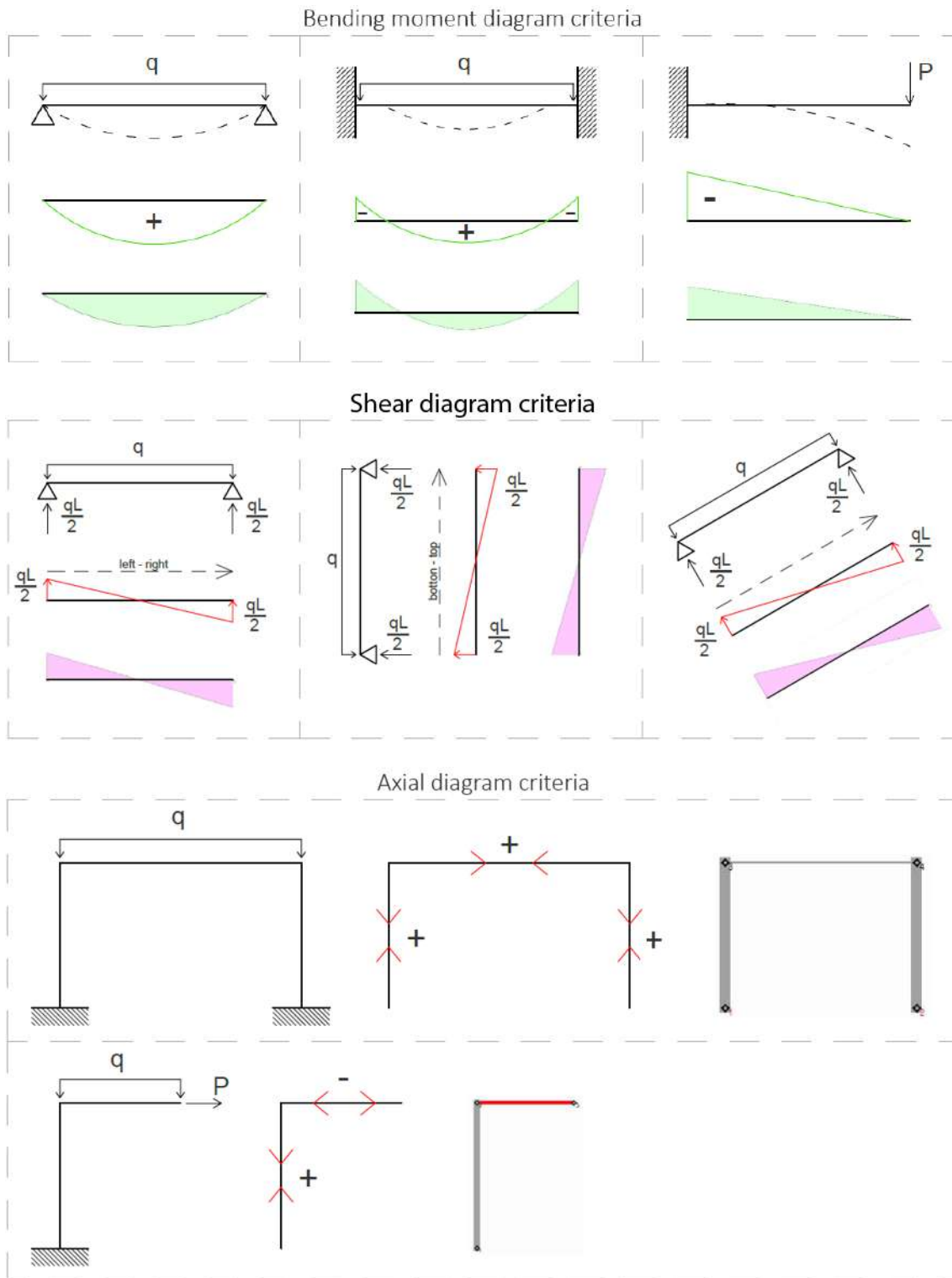
That's why, with this personal point of view, its content is mainly practical. In these notes we collect a reduced amount of theoretical concepts (which are nonetheless very important) exposed in a short and easy manner. And to consolidate these concepts, we provide a large amount of practical exercises.

You should understand this material as a handbook for my theoretical classes, because they have been conceived as a help for the student, complementing and completing the information received in class.

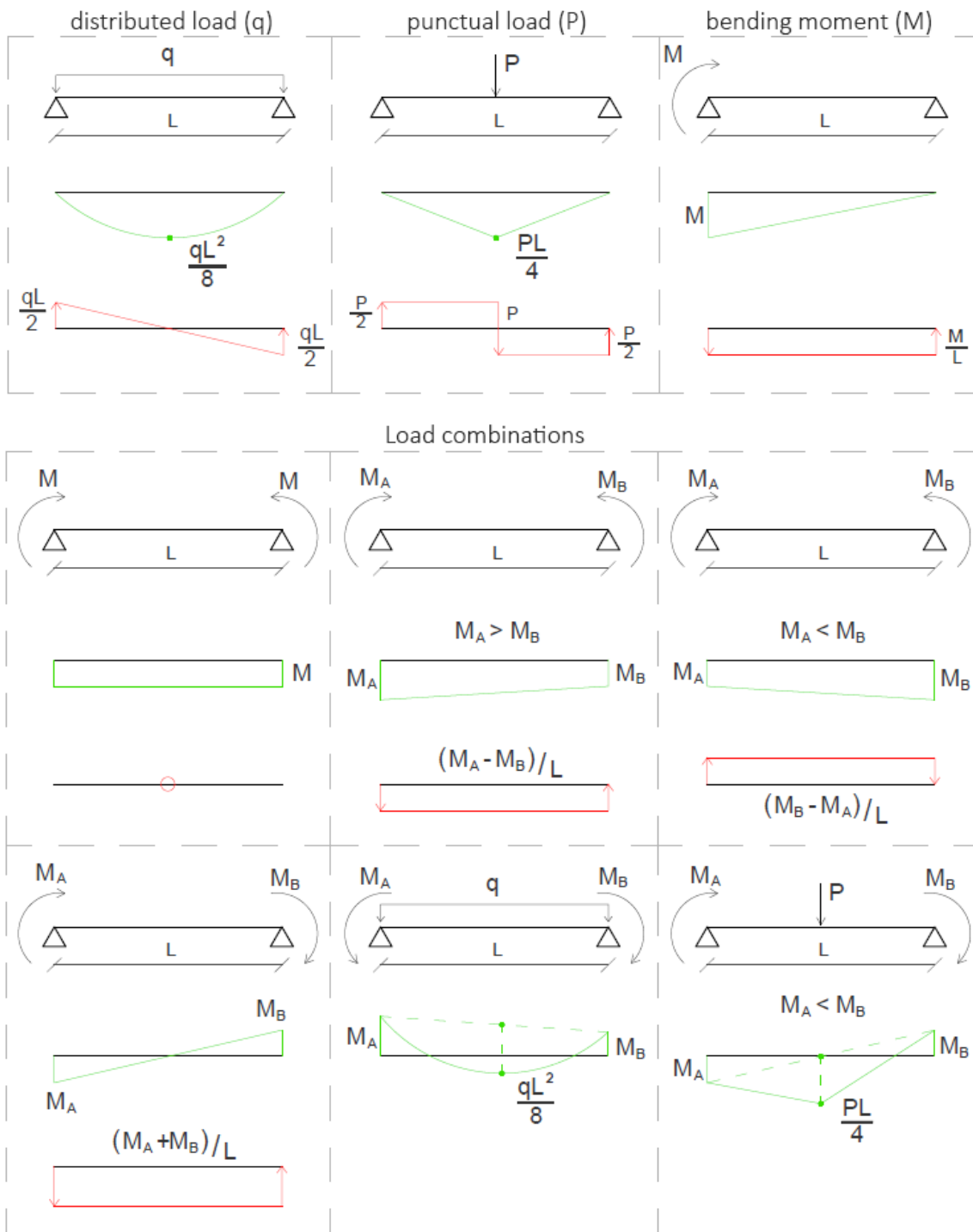
At the end of these notes we have a last chapter in which the main concepts we need to use matrix calculation software are exposed. There will be also a short tutorial with a practical example. In this way the student has a useful tool to check if the results of their calculations are ok.

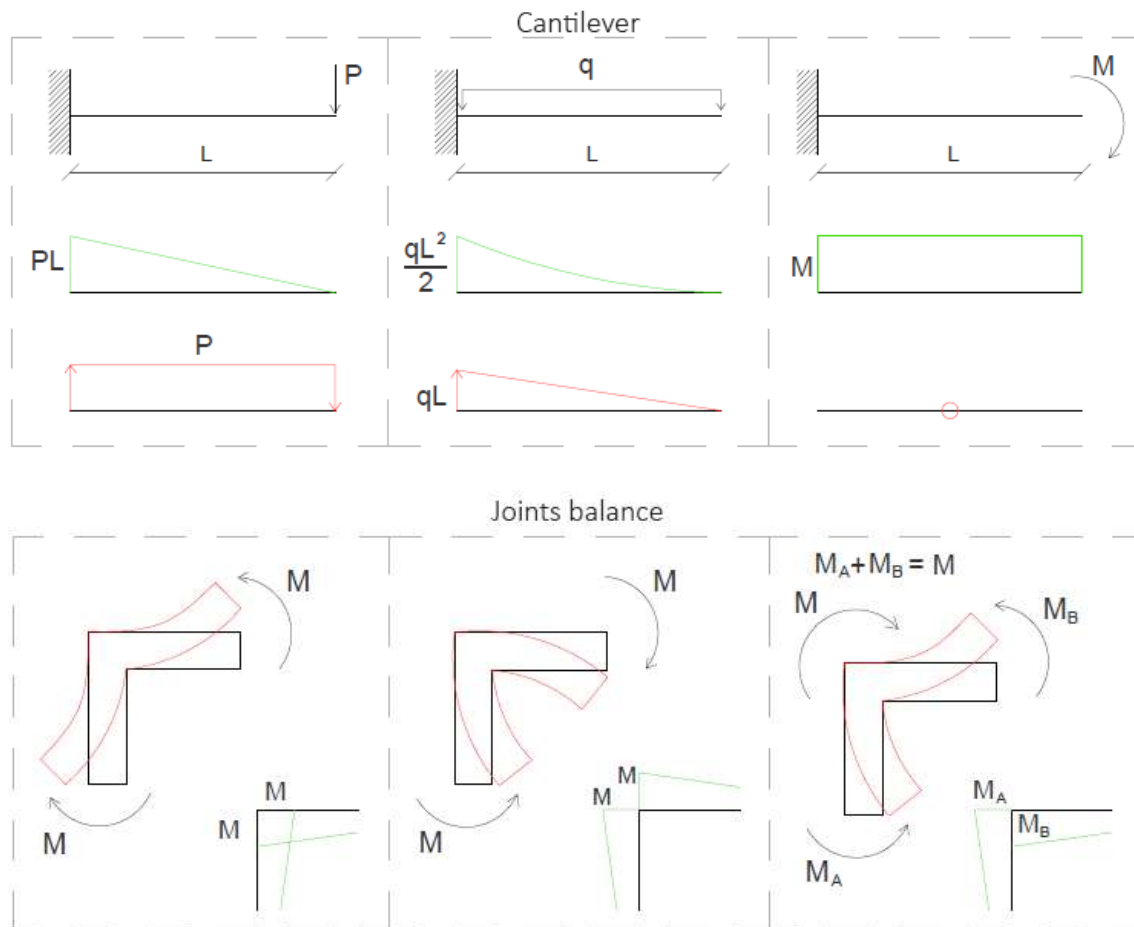
2. PREVIOUS CONCEPTS

2.1. BASIC CRITERIA FOR REPRESENTING DIAGRAMS

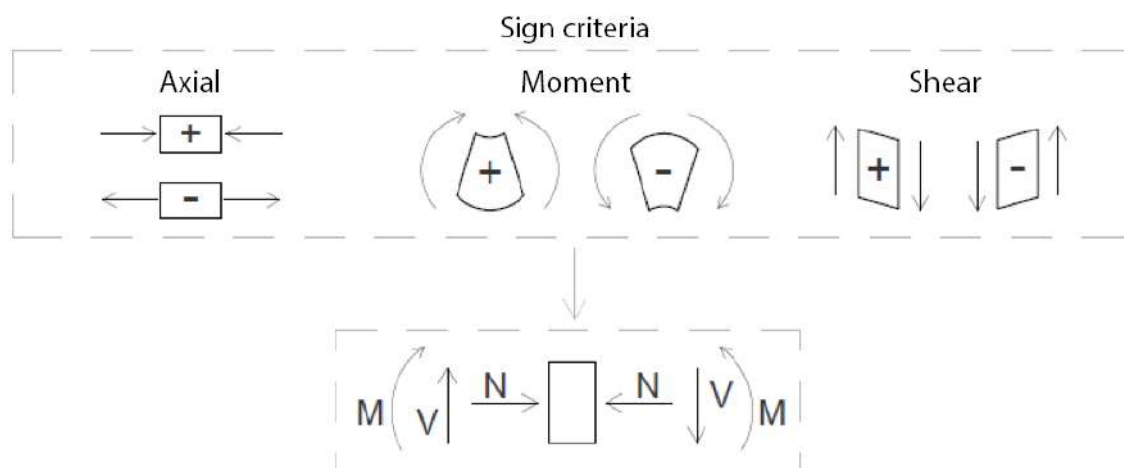


2.2. SUMMARY TABLE FOR REPRESENTING DIAGRAMS





2.3. STRENGTH SIGN CRITERIA

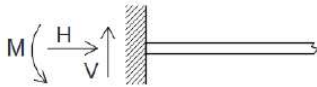


2.4. ISOSTATIC AND HYPERSTATIC STRUCTURES. BASIC CONCEPTS

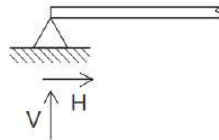
-Static equilibrium equation

$$\Sigma F_x = 0; \Sigma F_y = 0; \Sigma M_0 = 0$$

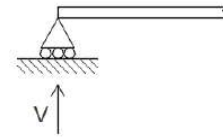
-Boundary conditions. Support types:



Freedom degree = 0
Reactions = 3
Fixed end



Freedom degree = 1
Reactions = 2
Hinged support



Freedom degree = 2
Reactions = 1
Roller support

-Isostatic structure:

- We can calculate the reactions using the equilibrium equations.
- Statically determinate

-Hyperstatic structures:

- We can't calculate the reactions using the equilibrium equations.
- Statically indeterminate

-Non-stable structure:

- Not in equilibrium.

Stability and indetermination degree

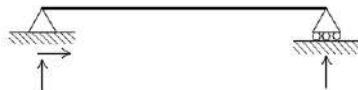
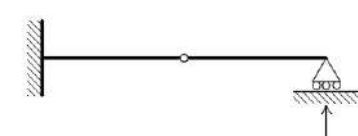
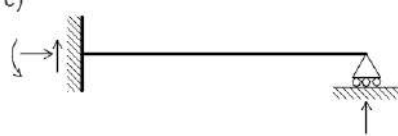
a. Beams

$$R < C + 3 \rightarrow \text{Non-stable}$$

$$R = C + 3 \rightarrow \text{Isostatic. Statically determinate}$$

$$R > C + 3 \rightarrow \text{Hyperstatic. Statically indeterminate}$$

R = reactions; C= additional conditions (ball joint $\rightarrow M=0$)

- a)  $R = 3; C = 0$
 $R = C + 3; 3 = 0 + 3 \rightarrow$ Isostatic
- b)  $R = 4; C = 1$
 $R = C + 3; 4 = 1 + 3 \rightarrow$ Isostatic
- c)  $R = 4; C = 0$
 $R > C + 3; 4 > 0 + 3 \rightarrow$ Hyperstatic;
Indetermination degree = 1

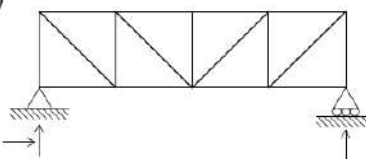
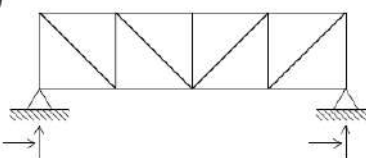
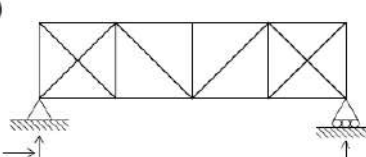
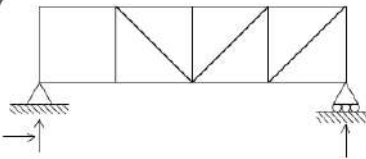
b. Trusses

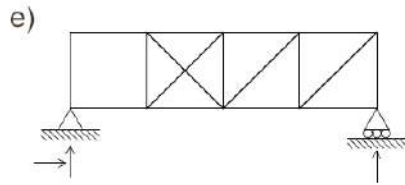
$B + R < 2J \rightarrow$ Non-stable

$B + R = 2J \rightarrow$ Isostatic. Determinate

$B + R > 2J \rightarrow$ Hyperstatic. Indeterminate

B= bars; R= reactions; J= joints

- a)  $R = 3; B = 17; J = 10$
 $B + R = 2J; 17 + 3 = 20 \rightarrow$ Isostatic
- b)  $R = 4; B = 17; J = 10$
 $B + R > 2J; 17 + 4 > 20 \rightarrow$ Hyperstatic
Indetermination degree = 1, Outer-Hyperstatic
- c)  $R = 3; B = 23; J = 12$
 $B + R > 2J; 23 + 3 > 24 \rightarrow$ Hyperstatic
Indetermination degree = 2, Inner-Hyperstatic
- d)  $R = 3; B = 16; J = 10$
 $B + R < 2J; 16 + 3 < 20 \rightarrow$ Non-stable



$$R = 3; B = 19; J = 11$$

$$B + R = 2J; 19 + 3 = 22 \rightarrow \text{Careful! Non-stable}$$

This condition is mandatory, but is not enough to have a stable structure.

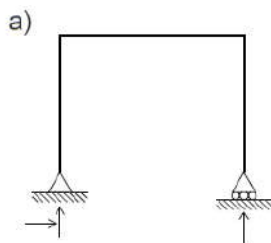
c. Frames

$$3B + R < 3J + C \rightarrow \text{Non-stable}$$

$$3B + R = 3J + C \rightarrow \text{Isostatic. Determinate}$$

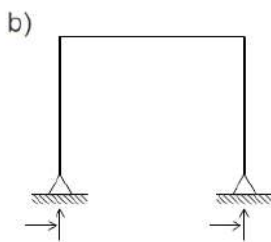
$$3B + R > 3J + C \rightarrow \text{Hyperstatic. Indeterminate}$$

B= bars; R= reactions; J= joints; C= additional conditions



$$R = 3; B = 3; J = 4$$

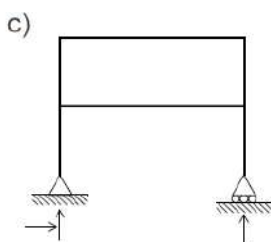
$$3B + R = 3J; 9 + 3 = 12 \rightarrow \text{Isostatic}$$



$$R = 4; B = 3; J = 4$$

$$3B + R > 3J; 9 + 4 > 12 \rightarrow \text{Hyperstatic}$$

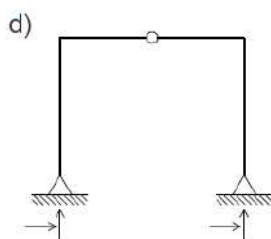
Indetermination degree = 1, outer-hyperstatic



$$R = 3; B = 6; J = 6$$

$$3B + R > 3J; 18 + 3 > 18 \rightarrow \text{Hyperstatic}$$

Indetermination degree = 3, inner-hyperstatic



$$R = 4; B = 4; J = 5; C = 1$$

$$3B + R = 3J + C; 12 + 4 = 15 + 1 \rightarrow \text{Isostatic}$$

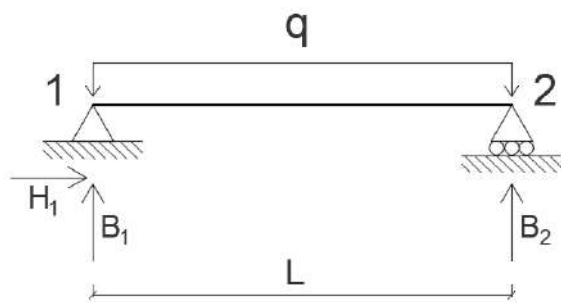
3. RESOLUTION OF ISOSTATIC FRAMES

3.1. CALCULATION OF REACTIONS

3.1.1. Overall equilibrium. Boundary conditions

On any structure, actions and reactions must be balanced. Therefore, we have to analyse the amount of reactions our structure has, taking into account the boundary conditions, and check if actions and reactions fulfil the static equilibrium equations. When calculating the equations, reactions are unknown. If the static equilibrium equations are enough to obtain the value of reactions, it means that the structure is isostatic.

Example



$$\Sigma B = 0 ; H_1 = 0$$

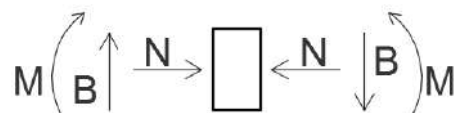
$$\Sigma B = 0 ; B_1 + B_2 = q \cdot l$$

$$\Sigma M_1 = 0 ; B_2 \cdot l - \frac{q \cdot l^2}{2} = 0 ; B_2 = \frac{q \cdot l}{2}$$

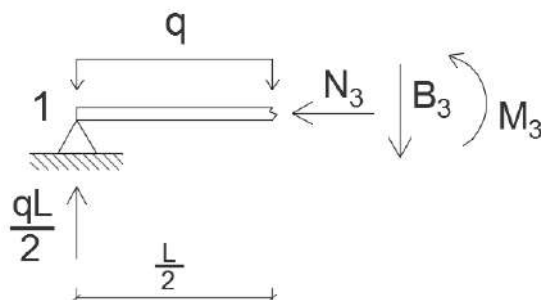
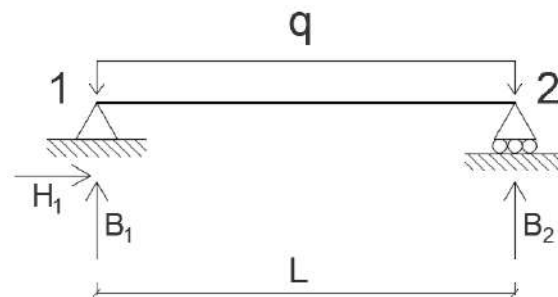
3.1.2. Partial equilibrium. Internal Strengths.

On a balanced structure, any isolated part will also be in equilibrium. If there is an overall equilibrium, there should also be a partial equilibrium. This means that if we split the structure in any part, the partial equilibrium will be maintained.

To analyse the partial equilibrium we will have to take into account the internal strengths:



Example



Strengths on the 3. Point section:

$$\Sigma H = 0 ; N_3 = 0$$

$$\Sigma B = 0 ; \frac{q \cdot L}{2} = q \frac{L}{2} + V_3 ; V_3 = 0$$

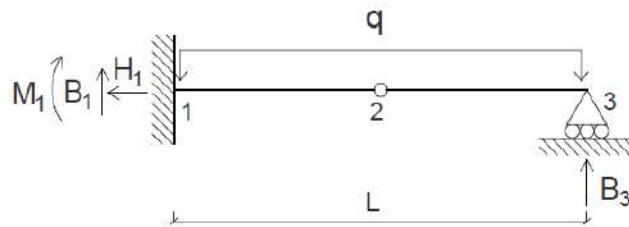
$$\Sigma M_3 = 0 ; M_3 + q \frac{L}{2} \cdot \frac{L}{4} - \frac{q \cdot L}{2} \cdot \frac{L}{2} = 0$$

$$M_3 = \frac{q \cdot L^2}{4} - \frac{q \cdot L^2}{8} = \frac{q \cdot L^2}{8}$$

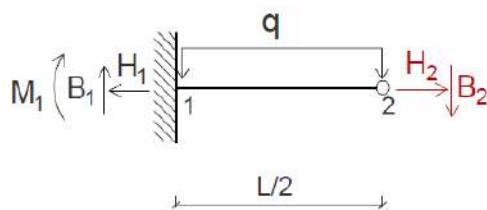
3.1.3. Bond conditions. Use of ball joints.

Ball joints are specific kinds of joints where bender moment is null. We can use that to solve structures that have more than three reactions. When analysing the equilibrium, we will be able to split the structure where the ball joints are located. All of these parts need also to conserve the inner balance, so we will have to check the partial equilibrium as we did in the previous section. With each splitting of the structure we have three equilibrium equations, but only two internal strengths since the moment is null. That's why ball joints reduce the hyperstaticity degree.

Example



PART 1



$$\Sigma H = 0 ; H_1 = H_2 = 0$$

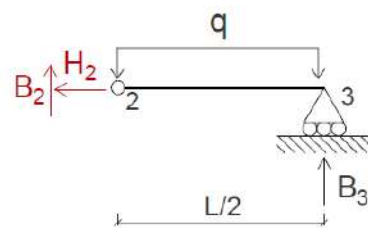
$$\Sigma B = 0 ; B_1 = q \cdot \frac{l}{2} + B_2 ;$$

$$B_1 = \frac{q \cdot l}{2} + \frac{q \cdot l}{4} = \frac{3 q \cdot l}{4}$$

$$\Sigma M_1 = 0 ; -\frac{q \cdot l^2}{8} - B_2 \cdot \frac{l}{2} - M_1 = 0 ;$$

$$M_1 = -\frac{q \cdot l^2}{8} - \frac{q \cdot l}{4} \cdot \frac{l}{2} = -\frac{q \cdot l^2}{4}$$

PART 2



$$\Sigma H = 0 ; H_2 = 0$$

$$\Sigma B = 0 ; B_3 + B_2 = q \cdot \frac{l}{2} ;$$

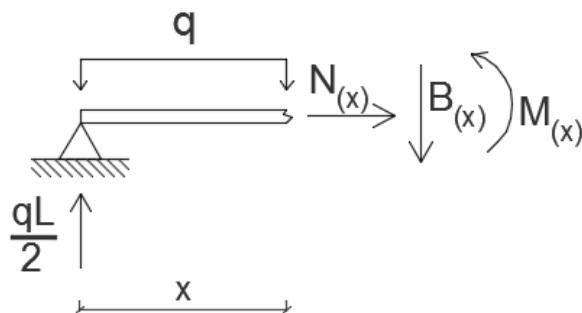
$$B_2 = \frac{q \cdot l}{4}$$

$$\Sigma M_2 = 0 ; B_3 \cdot \frac{l}{2} - \frac{q \cdot l^2}{8} = 0 ;$$

$$B_3 = \frac{q \cdot l}{4}$$

3.1.4. Graphic expression of the Strengths: diagrams of strengths.

If we split a bar in an unspecific X section, and we develop the equilibrium equations, internal strength will depend on X. Solving these equations for the internal strengths, we have graphics representing the value of the strengths for each point of the bar. We call these mathematical graphics diagrams of strengths.



Strengths on any section:

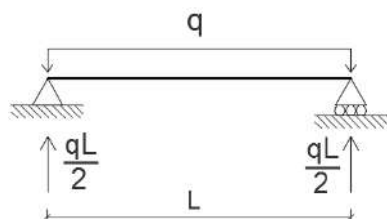
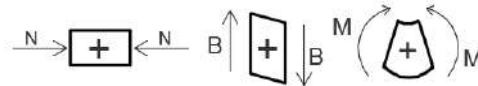
$$\Sigma H = 0 ; N_{(x)} = 0$$

$$\Sigma B = 0 ; \frac{q \cdot L}{2} = q \cdot x + V_{(x)} ; V_{(x)} = \frac{q \cdot L}{2} - q \cdot x$$

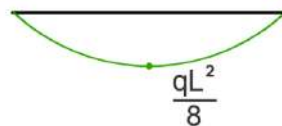
$$\Sigma M_3 = 0 ; -\frac{q \cdot L}{2} \cdot x + q \cdot x \cdot \frac{x}{2} + M_{(x)} = 0 ;$$

$$M_{(x)} = \frac{q \cdot L}{2} \cdot x - \frac{q \cdot x^2}{2}$$

Strength's sign criteria:



$$V_{(x)} = \frac{q \cdot L}{2} - q \cdot x$$



$$M_{(x)} = \frac{q \cdot L}{2} \cdot x - \frac{q \cdot x^2}{2}$$

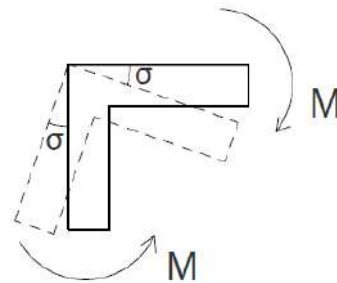
3.2. CALCULATION OF STRENGTHS IN THE JOINTS. SECTIONS

For drawing frame's strength diagrams, we need to calculate the inner strength in the nodes. For that, we have to split the structure in the section of the joint, and work out the partial equilibrium.

Depending on the kind of node, the existing strengths and the boundary conditions can vary. These are the main type of structural joints and their characteristics:

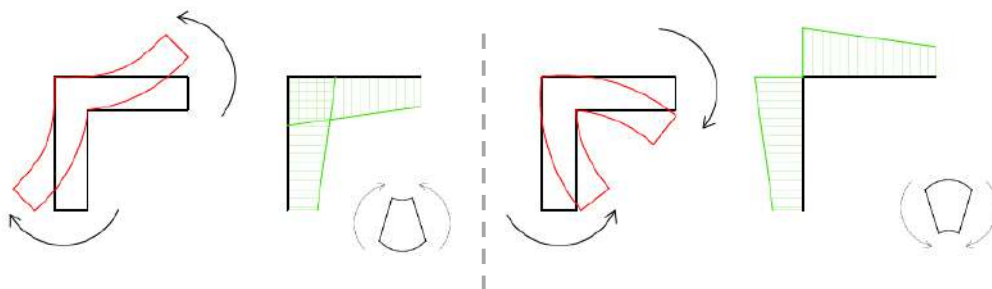
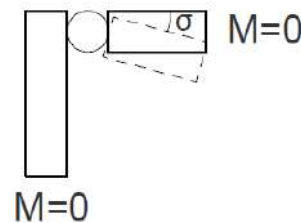
Stiff joints

- The rotated angle (σ) is the same in every bar.
- The bending moment is transmitted.



Ball joint (articulation)

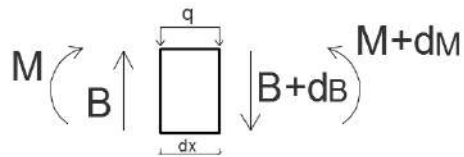
- The rotated angle (σ) can be different.
- There is no bending moment



3.3. STRENGTH DIAGRAMS

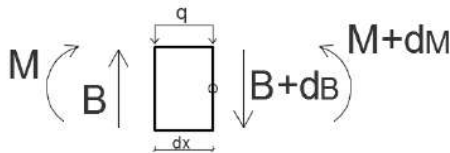
3.3.1. Mathematical relationship between strength diagrams

If we study the equilibrium equations of a differential element, we can infer some conclusions about the relationship between diagrams:



$$\Sigma B = 0 ;$$

$$B = q \cdot dx + B + dB ; \frac{dB}{dx} = -q$$



$$\Sigma M_o = 0 ;$$

$$B \cdot dx + M - q \cdot dx \cdot \frac{dx}{2} - M - dM = 0$$

$$B = \frac{dM}{dx} + \underbrace{\frac{q}{2} dx}_{dx \rightarrow 0} ; B = \frac{dM}{dx}$$

$$dx \rightarrow 0$$

$$\frac{q}{2} \cdot dx = 0$$

Therefore:

1- The derivative of the shear strength on any section, is the load supported, but changing the sign

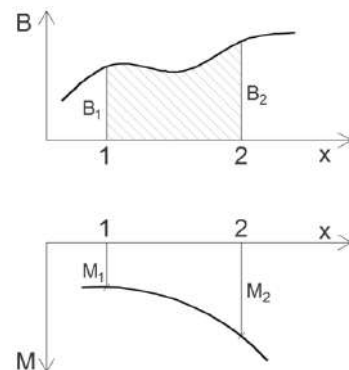
2- The derivative of the bending strength on any section, is the supported shear strength.

Consequences:

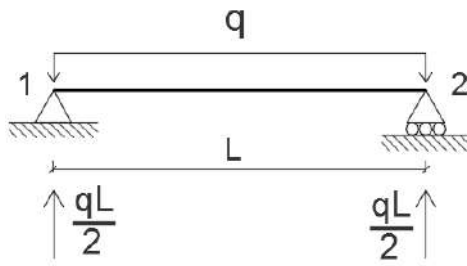
$$1) \quad V = 0 \rightarrow M_{MAX}$$

$$2) \quad M(x) = \int V(x) \cdot dx$$

$$3) \quad M_2 - M_1 = \int_1^2 V(x) \cdot dx$$



Example



$$V(x) = \frac{q \cdot L}{2} - q \cdot x$$

$$M(x) = \int V(x) \cdot dx ; M(x) = \int \left(\frac{q \cdot L}{2} - qx \right) \cdot dx ; M(x) = \frac{q \cdot L}{2} x - q x^2 \cdot \frac{1}{2} + C$$

Boundary conditions $\rightarrow M(0) = 0$

$$M(0) = \frac{q \cdot L}{2} \cdot 0 - \frac{q}{2} \cdot 0^2 + C ; \boxed{C = 0}$$

$$\boxed{V(x) = \frac{q \cdot L}{2} - q \cdot x}$$

$$\boxed{M(x) = \frac{q \cdot L}{2} x - \frac{q \cdot x^2}{2}}$$

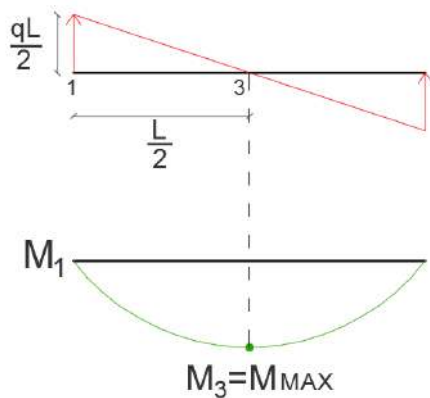
Maximum bending moment

$$V(x) = 0 \rightarrow M_{\text{MAX}}$$

$$V(x) = \frac{q \cdot L}{2} - q \cdot x = 0 ; \frac{q \cdot L}{2} = q \cdot x ; x = \frac{L}{2}$$

$$M\left(\frac{L}{2}\right) = M_{\text{MAX}} = \frac{q \cdot L}{2} \cdot \frac{L}{2} - \frac{q \cdot \left(\frac{L}{2}\right)^2}{2} = \frac{q \cdot L^2}{4} - \frac{q \cdot L^2}{8} = \frac{q \cdot L^2}{8}$$

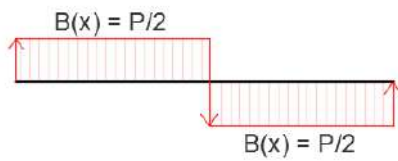
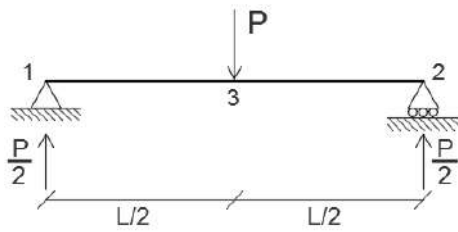
Using the area of the shear strength diagram



$$M_{\text{MAX}} - M_1 = \int_0^{\frac{L}{2}} V(x) \cdot dx = \Delta_{13}$$

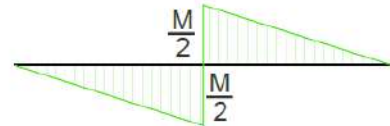
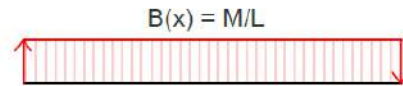
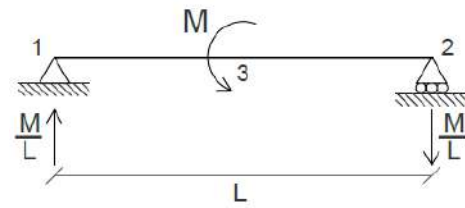
$$M_{\text{MAX}} - 0 = \frac{q \cdot L}{2} \cdot \frac{L}{2} \cdot \frac{1}{2} ; \boxed{M_{\text{MAX}} = \frac{q \cdot L^2}{8}}$$

3.3.2. Diagram representation. Other examples.

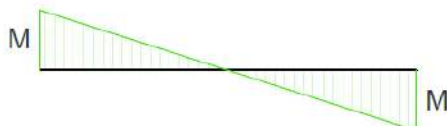
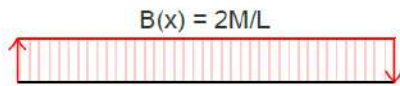
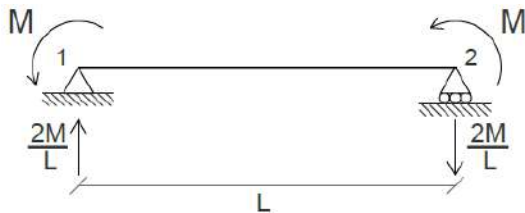


$$M(x) = \frac{P}{2}x \left(\frac{L}{2} \right); M(x) = -\frac{P}{2}x + \frac{PL}{2} \left(\frac{L}{2} \right)$$

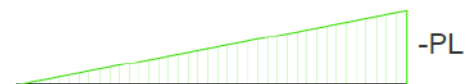
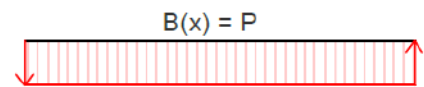
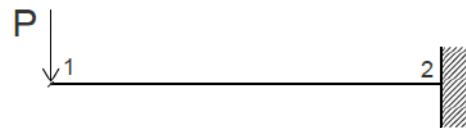
$$M_3 = M_{MAX}; M_{MAX} - M_1 = \Delta_{13}; M_{MAX} = \frac{P}{2} \cdot \frac{L}{2} = \frac{P \cdot L}{4}$$



$$M(x) = \frac{M}{L}x \left(\frac{L}{2} \right); M(x) = \frac{M}{L}x - M \left(\frac{L}{2} \right)$$



$$M(x) = \frac{2M}{L}x - M$$



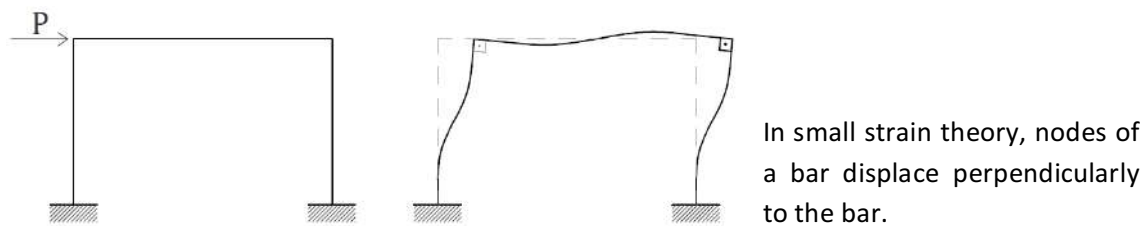
$$M(x) = -Px$$

3.4. REPRESENTING THE DEFORMATION OF THE FRAME

The following sections represent some basic rules we have to respect when drawing frame's deformation.

3.4.1. Non axial deformation hypothesis

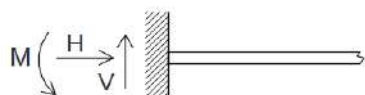
In frames working under bending moment strength, we can consider that the axial deformation is null. It means that the length of the bars doesn't change.



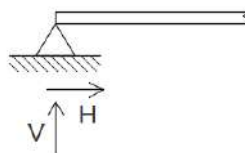
3.4.2. Boundary and bond conditions

The drawing of the deformation has to represent the boundary and bond conditions.

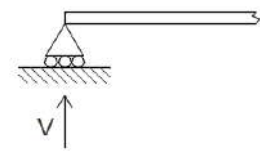
Boundary conditions:



Rotation and
translation restrained

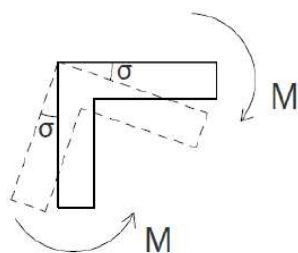


Free rotation



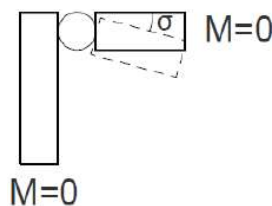
Free rotation and
1 translation

Bond conditions:



Stiff node

-The rotated angle (σ) is
the same.

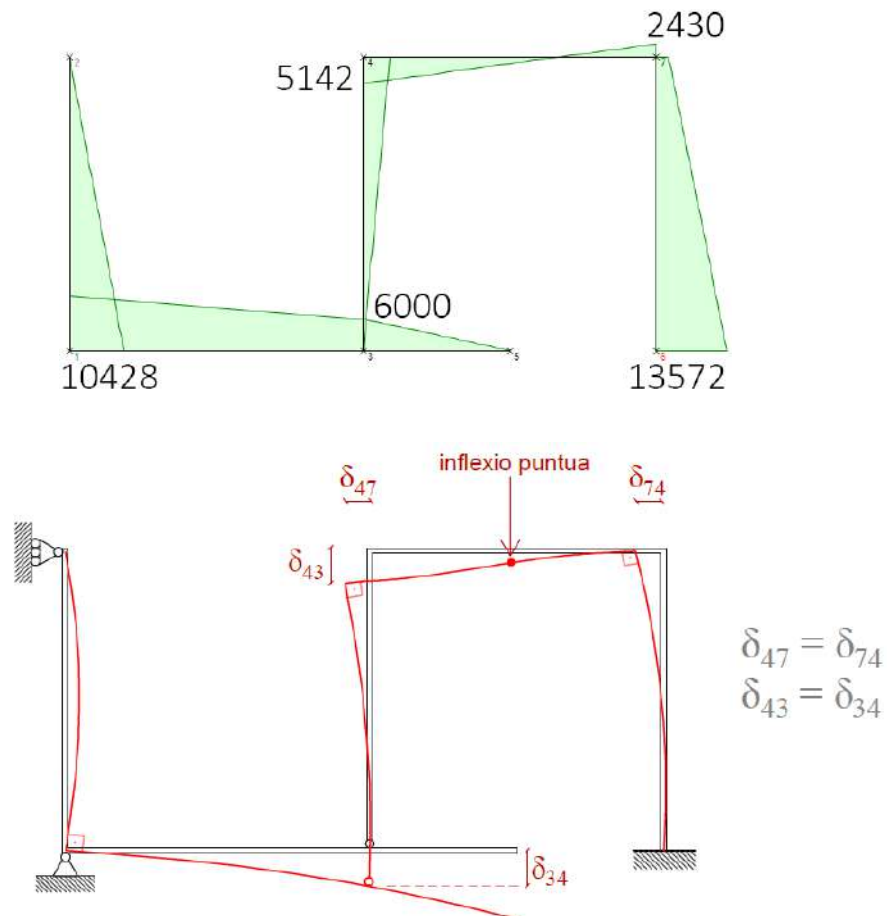


Hinge

-Free rotation of each
bar.

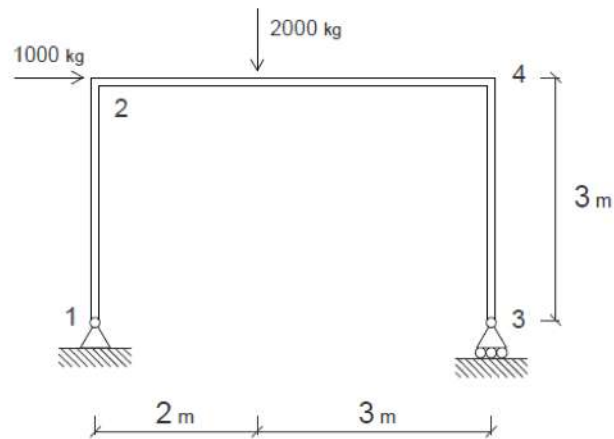
3.4.3. Curvature of the bending moment diagram

The drawing of the deformation has to represent the curvature which that bending moment diagram implies.

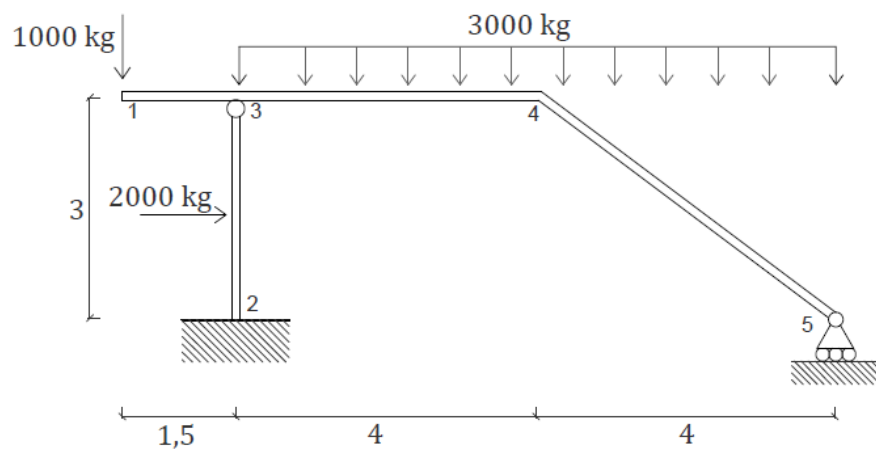


3.5. EXERCISES

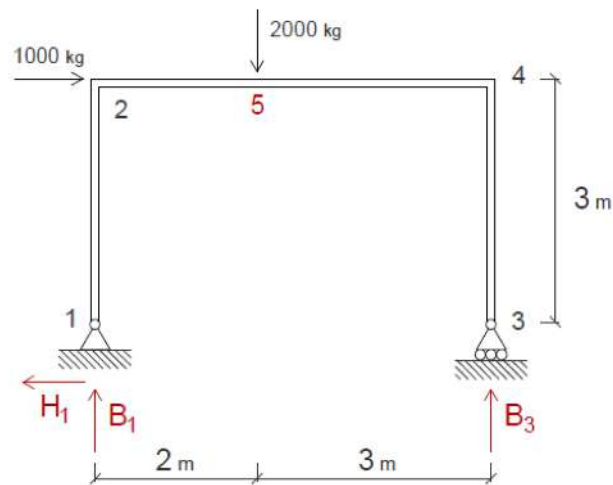
Exercise 1. Calculate the diagrams, deformations and reactions of the following structure.



Exercise 2. Calculate the diagrams, deformations and reactions of the following structure.



Exercise 1



$$\Sigma H_0 = 0 ; H_1 = 1000 \text{ kg}$$

$$\Sigma B_0 = 0 ; B_1 + B_3 = 2000 \text{ kg} ; \quad B_1 = 2000 - B_3 = 2000 - 1400 = 600 \text{ kg}$$

$$\Sigma M_1 = 0 ; 1000 \cdot 3 + 2000 \cdot 2 - B_3 \cdot 5 = 0 ; B_3 = 7000/5 = 1400 \text{ kg}$$

-Analysis of joint's sections:

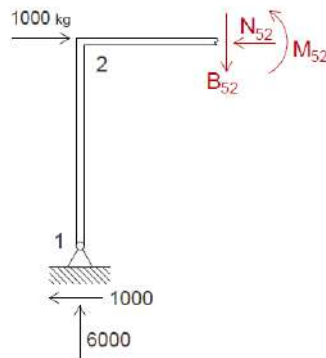


2-1 Section

$$B_{21} = 1000 \text{ kg}$$

$$N_{21} = 600 \text{ kg}$$

$$M_{21} = 3000 \text{ kg}\cdot\text{m}$$

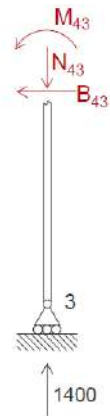


5-2 Section

$$B_{52} = 600 \text{ kg}$$

$$N_{52} = 0 \text{ kg}$$

$$M_{52} = 1000 \cdot 3 + 600 \cdot 2 = 4200 \text{ kg}\cdot\text{m}$$



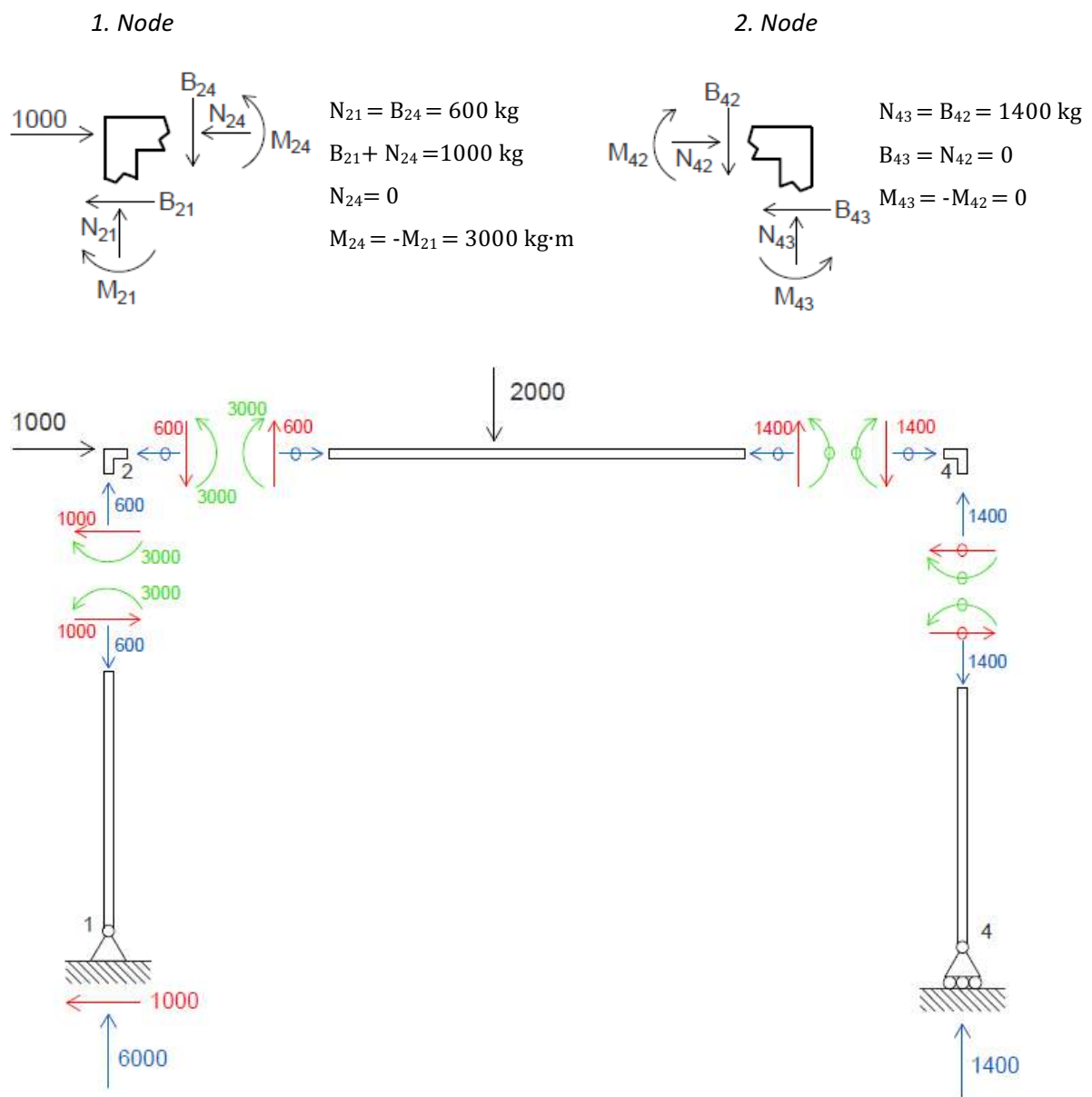
4-3 Section

$$B_{43} = 0 \text{ kg}$$

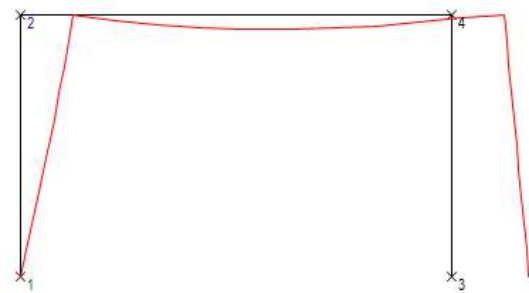
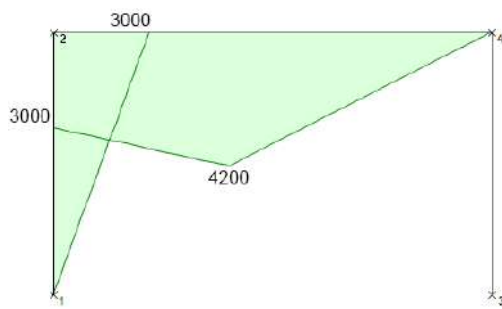
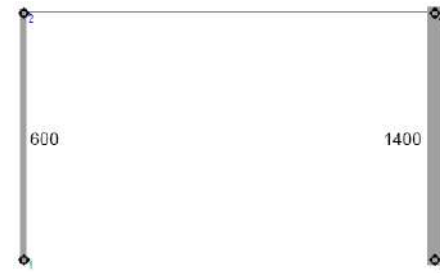
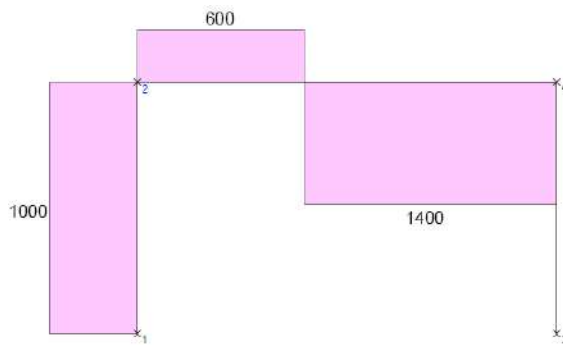
$$N_{43} = 1400 \text{ kg}$$

$$M_{43} = 0 \text{ kg}\cdot\text{m}$$

-Equilibrium in the nodes:

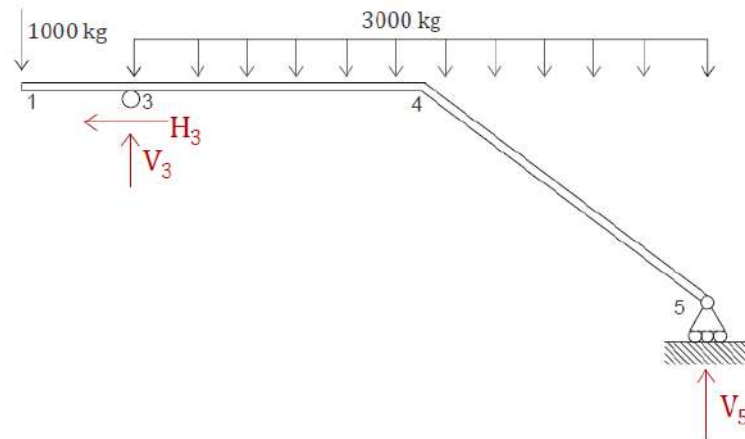


-Diagrams



Exercise 2

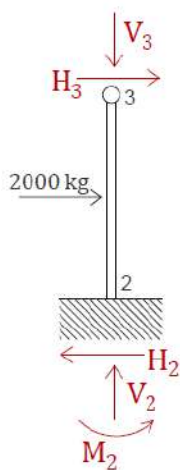
We separate the structure in two parts, where the ball joint is.



$$\Sigma H = 0 ; \boxed{H_3 = 0}$$

$$\Sigma V = 0 ; V_3 + V_5 = 1000 + 3000 \cdot 8 = 25000 \rightarrow \boxed{V_3 = 13187,5}$$

$$\Sigma M_3 = 0 ; 1000 \cdot 1,5 + V_5 \cdot 8 = 3000 \cdot 8 \cdot 4 ; 8 V_5 = 96000 - 1500 ; \boxed{V_5 = 11812,5}$$



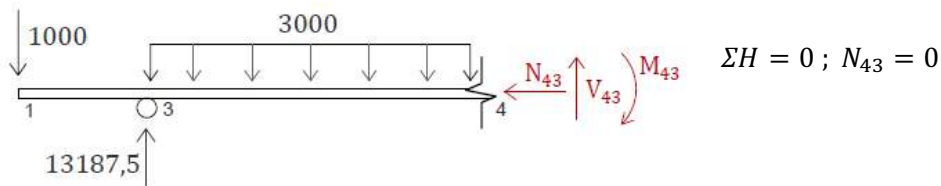
$$\Sigma H = 0 ; H_2 = H_3 + 2000 ; \boxed{H_2 = 2000}$$

$$\Sigma V = 0 ; V_2 = V_3 ; \boxed{V_2 = 13187,5}$$

$$\Sigma M_3 = 0$$

$$2000 \cdot 1,5 + M_2 = 3 \cdot H_2 ; 3000 + M_2 = 6000 ; \boxed{M_2 = 3000}$$

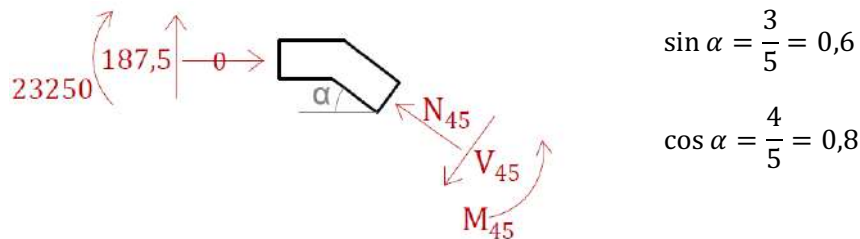
4-3 section



$$\Sigma V = 0; V_{43} + 13187,5 = 3000 \cdot 4 + 1000; V_{43} = -187,5 \text{ kg}$$

$$\Sigma M_4 = 0; 13187,5 \cdot 4 + M_{43} = 1000 \cdot 5,5 + 3000 \cdot 4 \cdot 2; M_{43} = -23250 \text{ kg} \cdot \text{m}$$

4. node equilibrium

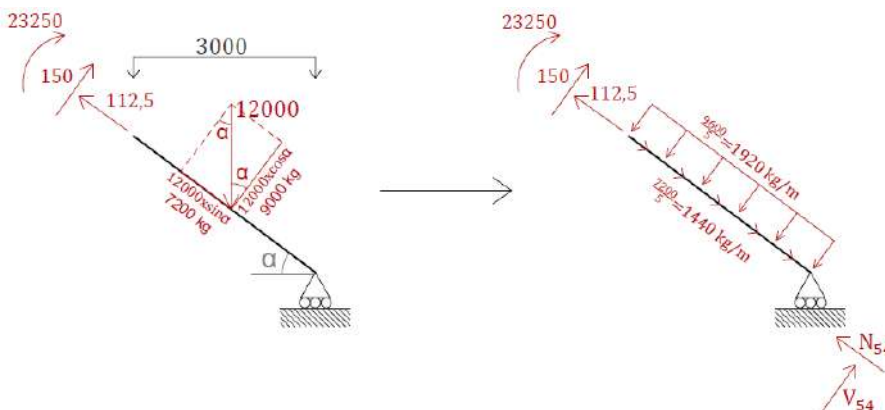


$$\Sigma M = 0; M_{45} = 23250$$

$$\Sigma H = 0; N_{45} \cdot 0,8 = -V_{45} \cdot 0,6; N_{45} = -0,75 V_{45}; N_{45} = -0,75 \cdot 150 = -112,5$$

$$\Sigma V = 0; V_{45} \cdot 0,8 - N_{45} \cdot 0,6 = 187,5; V_{45} \cdot 0,8 - (-0,75 V_{45}) \cdot 0,6 = 187,5; V_{45} = 150$$

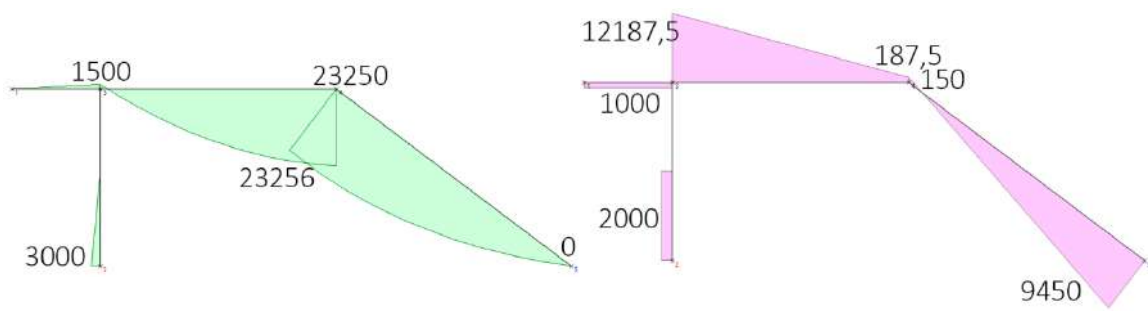
4-5 bar: research on the changing axial and shear strength



$$\Sigma N = 0; N_{54} = -112,5 + 1440 \cdot 5 = 7087,5 \text{ kg}$$

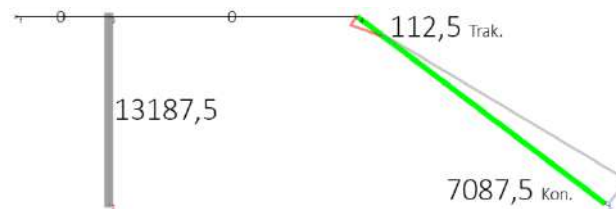
$$\Sigma V = 0; V_{54} = -150 + 1920 \cdot 5 = 9450 \text{ kg}$$

Diagrams

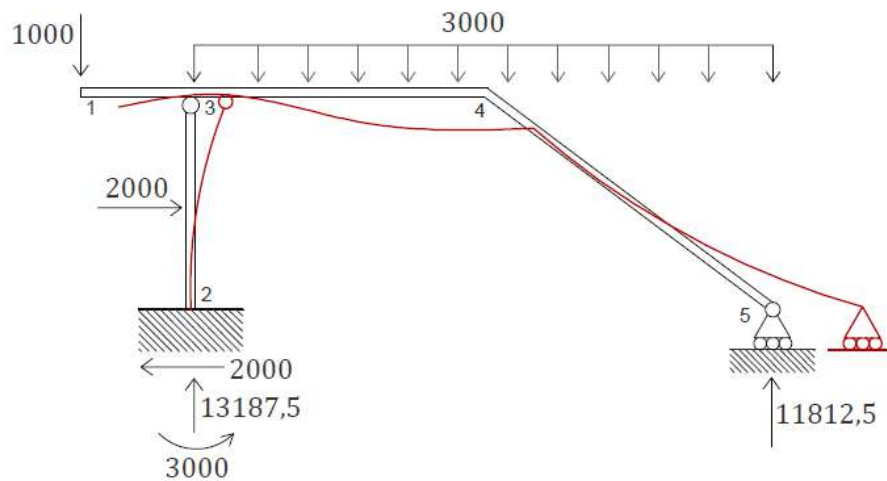


$$M_{MAX} - M_{45} = \Delta_{4-MAX} ; M_{MAX} = 25250 + (0,5 \cdot 150 \cdot 0,078)$$

$$M_{MAX} = 23250 + 5,85 \cong 23256 \text{ kg} \cdot \text{m}$$



Reactions and deformation



$$\Sigma H = 0 ; 2000 = 2000 \checkmark$$

$$\Sigma V = 0 ; 1000 + 3000 \cdot 8 = 13187,5 + 11812,5 ; 25000 = 25000 \checkmark$$

$$\Sigma M_2 = 0 ; 1000 \cdot 1,5 + 3000 + 11812,5 \cdot 8 = 2000 \cdot 1,5 + 3000 \cdot 8 \cdot 4$$

$$1500 + 3000 + 94500 = 3000 + 96000 ; 99000 = 99000 \checkmark$$

4. CALCULATION OF DEFORMATIONS

The classical methods of analysis of structures can be collected in two main families: Mathematical methods and Energetic methods. Among the numerous existing methods, in the following chapter we will study one of each family: The Elasticity Equation and the Unit Load method. Apart from that, in the first section we will review some basic formulations that represent the linear behaviour of structures under the action of different strengths.

4.1. RELATIONSHIP BETWEEN STRENGTH AND DEFORMATION

This first section is intended as a guideline for certain fundamental formulations of mechanics of materials, using the material's response to unidirectional stress under linear behaviour. The chapter will restrict itself to small-strain behaviour.

4.1.1. Deformation made by the axial strength

Hook's law:

$$\varepsilon = \frac{\Delta L}{L} = \text{Strain (proportional deformation)}$$

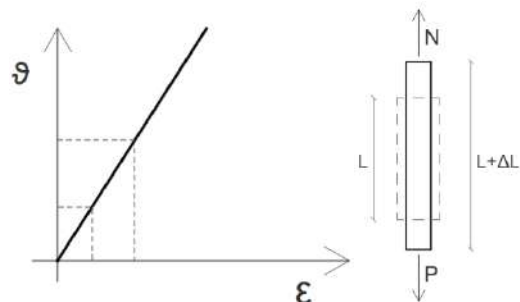
$$\vartheta = \frac{N}{A} = \text{Normal stress}$$

N = Axial strength

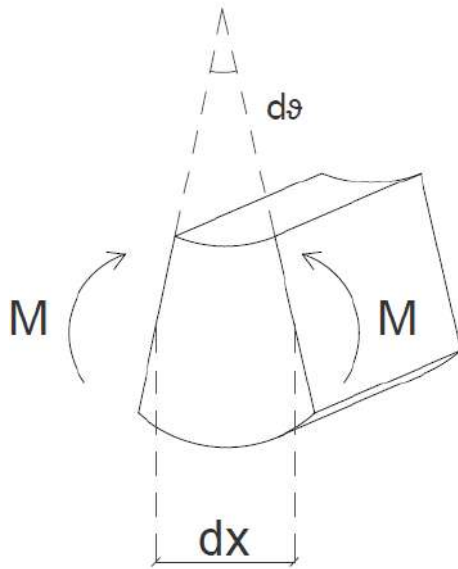
A = Section area

$$\boxed{E = \frac{\vartheta}{\varepsilon} = \text{Elastic modulus (Young's modulus)} = \text{constant}} \longrightarrow \text{Hooke's law}$$

$$E = \frac{\vartheta}{\varepsilon} \rightarrow E \frac{\frac{N}{A}}{\frac{\Delta L}{L}}; \frac{\Delta L}{L} = \frac{N}{AE}; \boxed{\Delta L = \frac{NL}{AE}} \longrightarrow \text{Deformation Length Measures}$$



4.1.2. Deformation made by the bending moment



Experimentally checked:

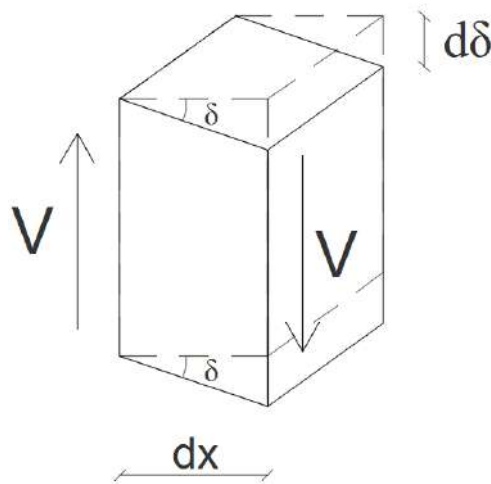
- Deflected sections remain straight.
- A proportional relationship exists between the rotated angle ($d\theta$), the analysed differential length (dx) and the bending moment:

$$KM = \frac{d\theta}{dx} \quad K: \text{Constant of proportionality}$$

$$K = \frac{1}{EI}; \quad \begin{array}{l} E: \text{Yield strength} \\ I: \text{Moment of inertia of the section} \\ \text{respective of the gravity center} \end{array}$$

$$\text{Then: } d\theta = \frac{M}{EI} \cdot dx; \quad \theta = \int_0^l \frac{M}{EI} dx$$

4.1.3. Deformation made by the shear strength



In an elastic behaviour, we observe a proportionality between $d\delta$ and the shear strength.

That proportionality is represented by the following expression:

$$\frac{d\delta}{dx} = V \cdot \frac{\kappa}{GA}$$

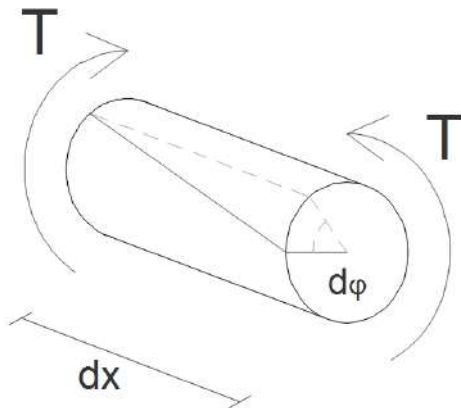
κ : factor dependent on the section shape

G : Indirect elasticity module

A : section area

$$\text{Then: } d\delta = \kappa \frac{V}{GA} \cdot dx; \quad \delta = \kappa \int_0^l \frac{V}{GA} dx$$

4.1.4. Deformation made by torsion



In this case, there is also a proportionality rule between the given strength and the resulting rotation:

$$\frac{d\varphi}{dx} = T \cdot \frac{1}{GI_T}$$

G : Shear modulus

I_T : Torsion inertia moment

Then:
$$d\varphi = \frac{T}{GI_T} \cdot dx; \quad \varphi = \int_0^l \frac{T}{GI_T} dx$$

4.2. ELASTICITY EQUATION. BEAM DEFORMATIONS

In structures working under axial strengths (trusses), the deformations are calculated by the axial strengths:

$$\Delta L = \frac{N \cdot L}{A \cdot E}$$

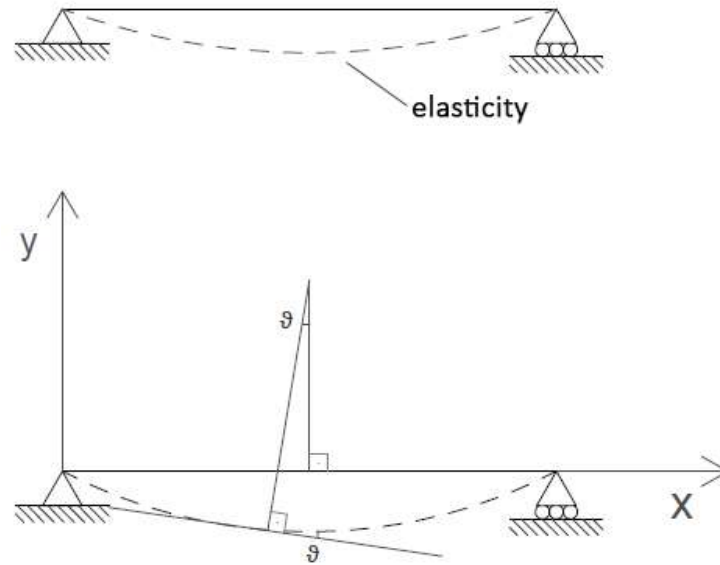
In the case of the beams, the deformation calculation is a little bit more complicated. We will have more than one strength at the same time, and moreover, these strengths can change along the bar.

To simplify this calculation, when calculating the beam deformation, we only take into account the impact of the bending moment, as the axial and shear strengths are negligible in comparison.

In this section, we will find the mathematical expression (the graphic) of the deformation curve, using the previous simplification.

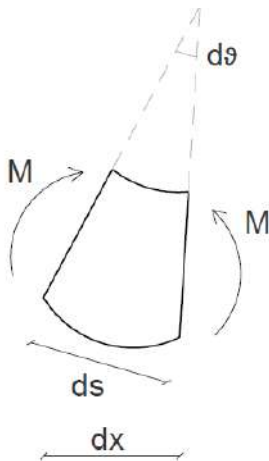
This curve will be called “elasticity equation”, and we will define it under “ x ” value: $y = f(x)$

Elasticity equation:



If we analyse the derivative of any point, we get the inclination of the tangent cord:

$$\vartheta = \frac{dy}{dx} \rightarrow \frac{d\vartheta}{dx} = \frac{d^2y}{dx^2}$$



At the same time, taking into account the formula in section 4.1.2, we know:

$$\frac{d\vartheta}{ds} = \frac{M}{EI}$$

In a differential element we can consider: $ds \approx dx$.

Then:

$$\frac{d\vartheta}{dx} = \frac{M}{EI}$$

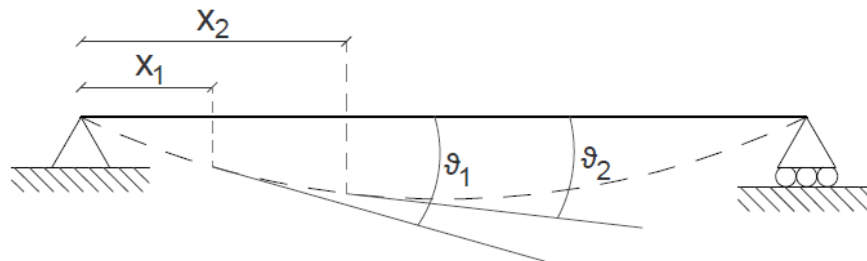
Attaching the previous two equations:

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Therefore, integrating twice the bending moment diagram, we get the elasticity equation. At the same time, the first integration of the moment diagram is the inclination on the curve tangent cord.

$$\frac{dy}{dx} = \vartheta$$

Now, let's have a look at the beam again:



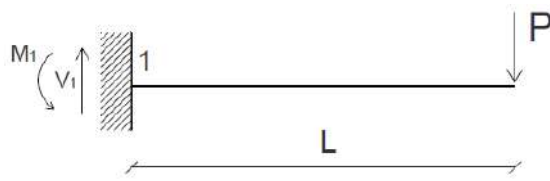
$$\begin{array}{l} x_2 > x_1 \rightarrow \Delta x \text{ positive} \\ \vartheta_2 < \vartheta_1 \rightarrow \Delta \vartheta \text{ negative} \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Then } \frac{d\vartheta}{dx} < 0$$

$$\frac{d\vartheta}{dx} = \frac{M}{EI} \quad \text{Then } \frac{-d\vartheta}{dx} = \frac{-M}{EI}$$

Following our criteria, when the deflection is downwards this has to be positive, as it is the natural deflection of the structures. Therefore, we should introduce a little change on the equation:

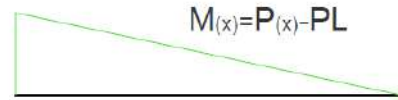
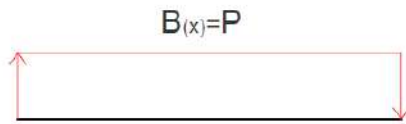
$$\boxed{\frac{d^2y}{dx^2} = \frac{-M}{EI}}$$

Example 1



$$\sum B = 0; B_1 = P$$

$$\sum M_1 = 0; M_1 = P \cdot L$$



$$\frac{d^2 y}{dx^2} = \frac{-M_{(x)}}{EI}$$

$$\frac{dy}{dx} = \int \frac{-(Px - PL)}{EI} dx; \frac{dy}{dx} = \frac{1}{EI} \left(\frac{-Px^2}{2} + Plx + C_1 \right) = \vartheta$$

$$y = \int \frac{1}{EI} \left(\frac{-Px^2}{2} + Plx + C_1 \right) dx = \frac{1}{EI} \left(\frac{-Px^3}{6} + \frac{Plx^2}{2} + C_1x + C_2 \right) = \delta$$

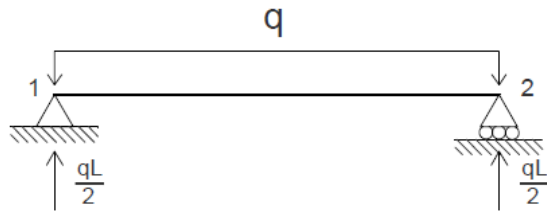
$$\vartheta_{(x)} = \frac{1}{EI} \left(\frac{-Px^2}{2} + Plx + C_1 \right); x = 0 \rightarrow \vartheta = 0; C_1 = 0$$

$$\delta_{(x)} = \frac{1}{EI} \left(\frac{-Px^3}{6} + \frac{Plx^2}{2} + C_1x + C \right); x = 0 \rightarrow \delta = 0; C = 0$$

Therefore:

$$\vartheta_{(x)} = \frac{1}{EI} \left(\frac{-Px^2}{2} + Plx \right) \quad ; \quad \delta_{(x)} = \frac{1}{EI} \left(\frac{-Px^3}{6} + \frac{Plx^2}{2} \right)$$

Example 2



Calculate:

- Maximum deflection
- Rotation on 1. support

$$B(x) = \frac{qL}{2} - qx$$

$$M(x) = \frac{qL}{2}x - \frac{qx^2}{2}$$

$$\frac{d^2y}{dx^2} = \frac{-M(x)}{EI}$$

$$\frac{dy}{dx} = \vartheta(x) = \int \frac{-M(x)}{EI} dx = \frac{1}{EI} \int \left(-\frac{ql}{2}x + \frac{qx^2}{2} \right) dx = \frac{1}{EI} \left(-\frac{ql}{2} \cdot \frac{x^2}{2} + \frac{qx^3}{2 \cdot 3} + C_1 \right)$$

$$y = \delta(x) = \int \frac{1}{EI} \left(-\frac{qlx^2}{4} + \frac{qx^3}{6} + C_1 \right) dx; \delta(x) = \frac{1}{EI} \left(-\frac{qlx^3}{12} + \frac{qx^4}{24} + C_1x + C_2 \right)$$

$\delta_{(0)} = 0; \delta_{(l)} = 0$ (Boundary conditions)

$$\delta_{(0)} = 0; \delta_{(0)} = \frac{1}{EI} \left(-\frac{ql \cdot 0}{12} + \frac{q \cdot 0}{24} + C_1 \cdot 0 + C_2 \right) = 0 \rightarrow C_2 = 0$$

$$\delta_{(0)} = 0; \delta_{(l)} = \frac{1}{EI} \left(-\frac{ql^4}{12} + \frac{ql^4}{24} + C_1l \right) = 0 \rightarrow C_1 = \frac{ql^3}{24}$$

Then

$$\vartheta(x) = \frac{1}{EI} \left(-\frac{ql \cdot x^2}{4} + \frac{qx^3}{6} + \frac{ql^3}{24} \right); \vartheta_{(0)} = \frac{ql^3}{24EI}$$

$$\delta(x) = \frac{1}{EI} \left(-\frac{ql \cdot x^3}{12} + \frac{qx^4}{24} + \frac{ql^3}{24}x \right); \delta_{\left(\frac{l}{2}\right)} = \frac{1}{EI} \left(-\frac{ql^4}{93} + \frac{ql^4}{384} + \frac{ql^4}{48} \right)$$

$$\boxed{\delta_{MAX} = \frac{5}{384} \frac{ql^4}{EI}}$$

4.3. ENERGETIC METHODS. UNIT LOAD

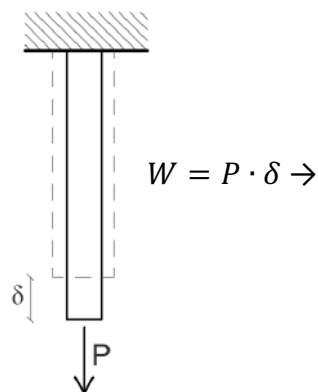
The energetic methods are a family of calculation procedures which use the equilibrium between the work of internal and external forces to deduce structural deformations. In these notes we will focus on the Unit Load method, due to its pedagogical value (it serves to practice the resolution of static structures) and the simplicity of its application. To understand this method we need to first expose some previous concepts: equilibrium of the work of internal and external forces, Virtual Work Principle and the Principle or Reciprocity.

4.3.1. Equilibrium between the work of internal and external forces

As we know, all the Work done becomes energy. As the systems we analyse are static systems they don't create any kinetic energy. We also assume that there is no inner friction (otherwise heat would be created). In this situation, all the Work done is going to become deformation potential-energy.

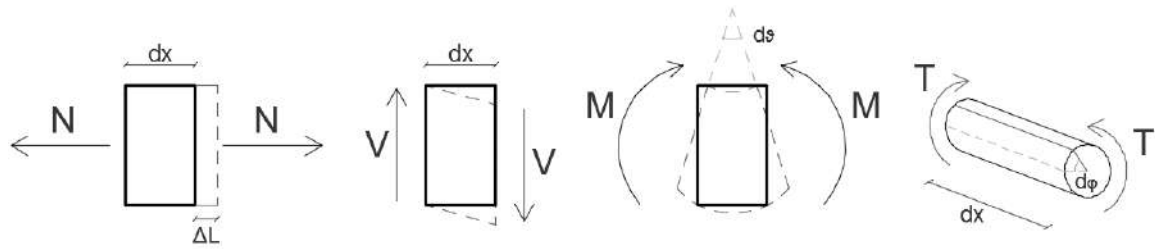
$$W_e(\text{external forces work}) = W_i(\text{internal forces work})$$

-External forces work (W_{EXT})



$$W_e = \int_0^\delta P \cdot d\delta = \int_0^\delta K \cdot \delta \cdot d\delta = \frac{K \cdot \delta^2}{2} = \frac{1}{2} \cdot p \cdot \delta$$

-Deformation energy or Work deformation of the internal strengths (W_i)



Using the previous explanation, the deformation energy or the work deformation created by the internal strengths is:

Axial: $W_i = N \cdot \Delta l ; \Delta l = \frac{N \cdot l}{AE} ; W_i = \frac{N \cdot N \cdot l}{AE} \quad (N=\text{constant})$

Shear: $W_i = V \cdot d\delta ; d\delta = \kappa \int \frac{V}{GA} \cdot dx ; W_i = \kappa \int_0^l \frac{V \cdot V}{GA} dx$

Flection: $W_i = M \cdot d\vartheta ; d\vartheta = \int \frac{M}{EI} \cdot dx ; W_i = \int_0^l \frac{M \cdot M}{EI} dx$

Torsion: $W_i = T \cdot d\varphi ; d\varphi = \int \frac{T}{GI_T} \cdot dx ; W_i = \int_0^l \frac{T \cdot T}{GI_T} dx$

Therefore, inner deformation Work's overall expression is:

$$W_i = \int_0^l \frac{N_x^2}{EI} dx + \kappa \int_0^l \frac{V_y^2}{GA} dx + \kappa \int_0^l \frac{V_z^2}{GA} dx + \int_0^l \frac{M_y^2}{EI} dx + \int_0^l \frac{M_z^2}{EI} dx + \int_0^l \frac{T_x^2}{EI} dx$$

Most of the structures that we are going to analyse are classified in two groups:

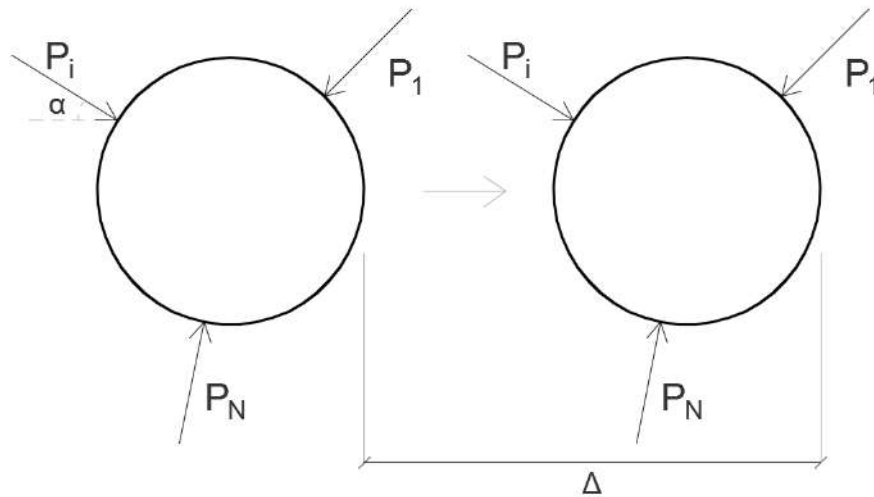
-Truss: $W_i = \sum \frac{N_i^2 l_i}{AE}$

-Frames*: $W_i = \int_0^l \frac{M^2}{EI} dx$

*When plane frames work on flection, usually there is no torsion, and the axial and shear strengths are negligible compared to the flection.

4.3.2. Virtual Work Principle

Let's analyse this rigid body:



When we introduce a virtual displacement Δ (we call it virtual as it's cause is not defined) on a system in balance ($\Sigma H=0$, $\Sigma B=0$) the following statement is true:

$$W = \sum_{i=1}^{i=n} P_i \cdot \Delta \cdot \cos \alpha = 0$$

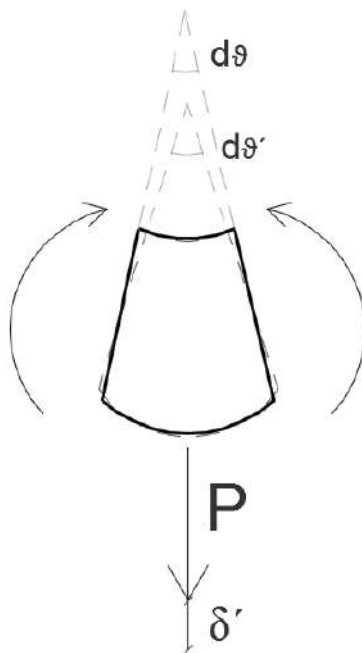
This statement is a direct consequence of the equilibrium requirements:

$$\Sigma H = 0; \sum_{i=1}^{i=n} P_i \cdot \cos \alpha = 0 \rightarrow \Delta \cdot \Sigma P_i \cdot \cos \alpha = 0$$

Therefore, the Virtual Work Principle is:

On a system in equilibrium, when a virtual displacement is introduced, the Work made by the system is null: $W=0$

When we study an elastic body, the system is balanced through internal deformations. This means that the principle is illustrated as follows:



$$W_{EXT} + W_{INT} = 0; P \cdot \delta' = M \cdot d\theta'$$

$$W_{EXT} = P \cdot \delta'$$

$$W_{INT} = -M \cdot d\theta' \text{ (Work of the internal strengths)}$$

Then, the Work made by the virtual displacement of the external forces, is in equilibrium with the Work created by the virtual deformation of the inner strengths.

In this situation, we analyse a differential element working on a flection strength. The shear and axial strengths are not taken into account as they are negligible.

Hence, the overall expression of the Virtual Work Principle is:

$$\Sigma P \cdot \delta' + \Sigma R \cdot \Delta' = \Sigma N \cdot \Delta l' + \int V \cdot d\delta' + \int M \cdot d\theta' + \int T \cdot d\varphi'$$

R: Reactions

Δ : Displacements on the reactions (if there is any)

4.3.3. Principle of Reciprocity

The Principle of Reciprocity is an evolution of the Virtual Work Principle. It is a necessary step to understanding to the Unit Load method.

Through the Virtual Work Principle, we have analysed a real system with a virtual displacement: $P, R, N, M, V, T; \delta', \Delta', dl', d\theta', d\delta', d\varphi'$

However, this principle can be done the other way round. We can apply a virtual force on a real deformation: $P', R', N', M', V', T'; \delta, \Delta, dl, d\theta, d\delta, d\varphi$

In this case:

$$\Sigma P' \cdot \delta + \Sigma R' \cdot \Delta = \Sigma N' \cdot \Delta l + \int V' \cdot d\delta + \int M' \cdot d\theta + \int T' \cdot d\varphi$$

These two cases developed:

$$\begin{aligned} \Sigma P' \cdot \delta + \Sigma R' \cdot \Delta &= \Sigma N' \cdot \frac{N \cdot l}{AE} + \mathcal{H} \int \frac{V' \cdot V}{GA} \cdot dx + \int \frac{M' \cdot M}{EI} \cdot dx + \int \frac{T' \cdot T}{GI_T} \cdot dx \\ \Sigma P \cdot \delta' + \Sigma R \cdot \Delta' &= \Sigma N \cdot \frac{N' \cdot l}{AE} + \mathcal{H} \int \frac{V \cdot V'}{GA} \cdot dx + \int \frac{M \cdot M'}{EI} \cdot dx + \int \frac{T \cdot T'}{GI_T} \cdot dx \end{aligned} \quad \left. \vphantom{\begin{aligned} \Sigma P' \cdot \delta + \Sigma R' \cdot \Delta &= \Sigma N' \cdot \frac{N \cdot l}{AE} + \mathcal{H} \int \frac{V' \cdot V}{GA} \cdot dx + \int \frac{M' \cdot M}{EI} \cdot dx + \int \frac{T' \cdot T}{GI_T} \cdot dx \\ \Sigma P \cdot \delta' + \Sigma R \cdot \Delta' &= \Sigma N \cdot \frac{N' \cdot l}{AE} + \mathcal{H} \int \frac{V \cdot V'}{GA} \cdot dx + \int \frac{M \cdot M'}{EI} \cdot dx + \int \frac{T \cdot T'}{GI_T} \cdot dx \end{aligned}} \right\} \text{Equals}$$

Therefore:

$$\Sigma P' \cdot \delta + \Sigma R' \cdot \Delta = \Sigma P \cdot \delta' + \Sigma R \cdot \Delta'$$

4.3.4. Unit Load method

This method is a specific application of the Principle of Reciprocity exposed in the previous section, where we apply a unit load as a virtual system. We use this method to calculate the displacements and the rotations in any point. For that, we apply a virtual unit force or a virtual unit moment in the location where we want to calculate a deformation. As a result, we should consider the expression of the previous section in this way:

For a deflection in a bar working under bending stress

$$\underbrace{1' \cdot \delta}_{\text{Principle of Recirpocity}} = \underbrace{1 \cdot \delta'}_{\text{Work of the Internal strength}} = M \cdot d\vartheta' = \int \frac{M \cdot M'}{EI} \cdot dx$$

Principle of
Recirpocity

Work of the
Internal
strength

Therefore:

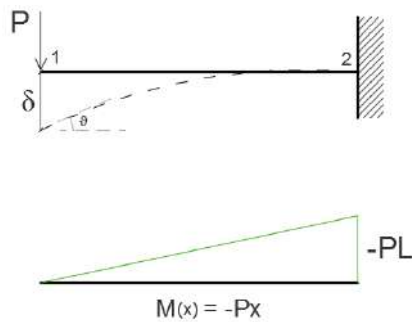
$$1' \cdot \delta = \delta = \int \frac{M \cdot M'}{EI} \cdot dx \quad (\text{displacement})$$

For a rotation in a bar working under bending stress

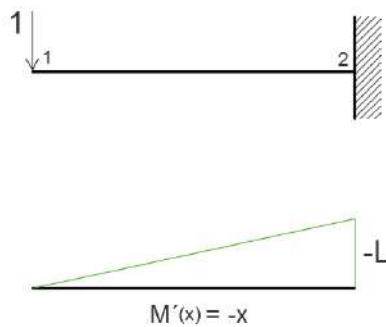
$$1' \cdot \vartheta = \vartheta = \int \frac{M \cdot M'}{EI} \cdot dx \quad (\text{rotation})$$

To better understand these two statements, let's see the following example:

Calculate the deflection and the rotation on point 1.



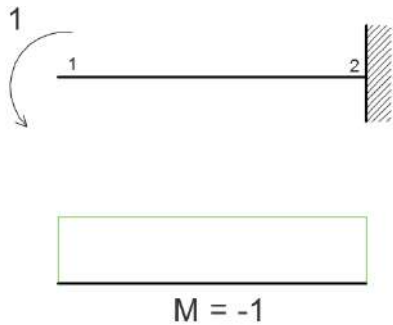
To calculate the deflection we have to apply a unit load to the point that we are analysing. That will be the virtual system:



Then:

$$1' \cdot \delta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx; \delta = \int_0^l \frac{(-Px) \cdot (-x)}{EI} \cdot dx = \int_0^l \frac{Px^2}{EI} \cdot dx = \frac{Px^3}{3EI} \Big|_0^l = \frac{Pl^3}{3EI}$$

To calculate the rotation we will apply a unit moment in the same point:



*In both cases, the positive sign in the result means that the deformation is done in the same direction as the unit load.

Then:

$$1' \cdot \vartheta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx = \int_0^l \frac{(-Px) \cdot (-1)}{EI} \cdot dx = \int_0^l \frac{Px}{EI} \cdot dx = \frac{Px^2}{2EI} \Big|_0^l = \frac{Pl^2}{2EI}$$

As we can see, when applying this method we always have to solve an integral that represent the multiplication of basic geometric graphics (bending moment graphics). To simplify this procedure, we can solve these basic integrals by using a table (Mohr Integrals Table. Appendix 1 of these notes). To understand how to use it, let's solve again the same example but by using the Mohr Integrals Table.

Deflection in point 1:

$$1' \cdot \delta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx; \quad M = -Px; \quad M' = -x$$

Real	
Unit load	$\int_0^l MM' dx = \frac{1}{3} \cdot M \cdot M' \cdot L = \frac{1}{3} \cdot -PL \cdot -L \cdot L = \frac{PL^3}{3}$

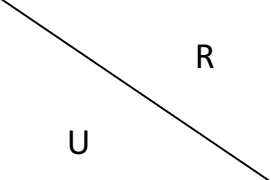
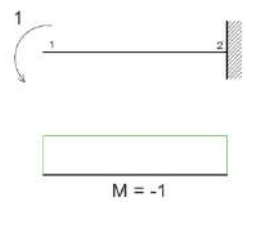
Therefore:

$$1' \cdot \delta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx = \frac{Pl^3}{3EI}$$

It should be noted that in the formulations of the Mohr Integrals Table, “M” represents the maximum moment of the real system, and “M’” represents the maximum moment of the virtual system.

Rotation in point 1:

$$1' \cdot \vartheta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx; \quad M = -Px; \quad M' = -1$$

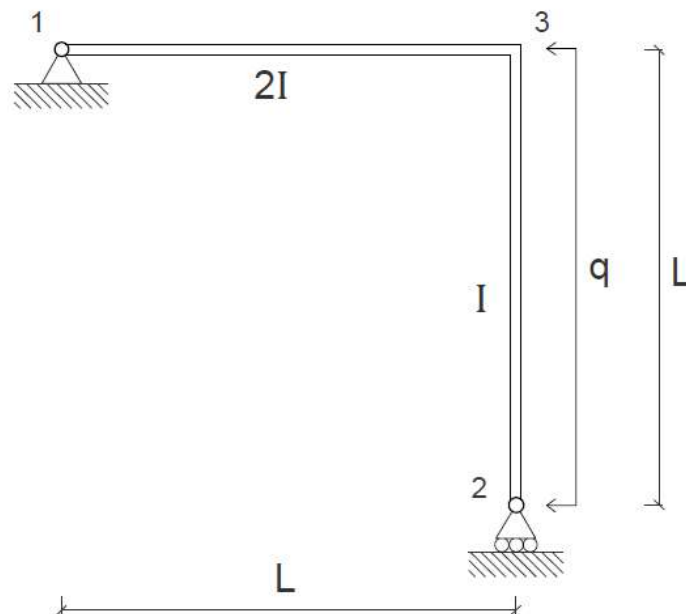
U  R	 
 M = -1	$\int_0^l MM' dx = \frac{1}{2} \cdot M \cdot M' \cdot L = \frac{1}{2} \cdot -PL \cdot -1 \cdot L = \frac{PL^2}{2}$

Therefore:

$$1' \cdot \vartheta = \int_0^l \frac{M \cdot M'}{EI} \cdot dx = \frac{Pl^2}{2EI}$$

4.3.5. Unit load method. Exercises with isostatic frames

EXERCISE 1. Calculate, using the unit load method, the horizontal displacement of the 2. support.

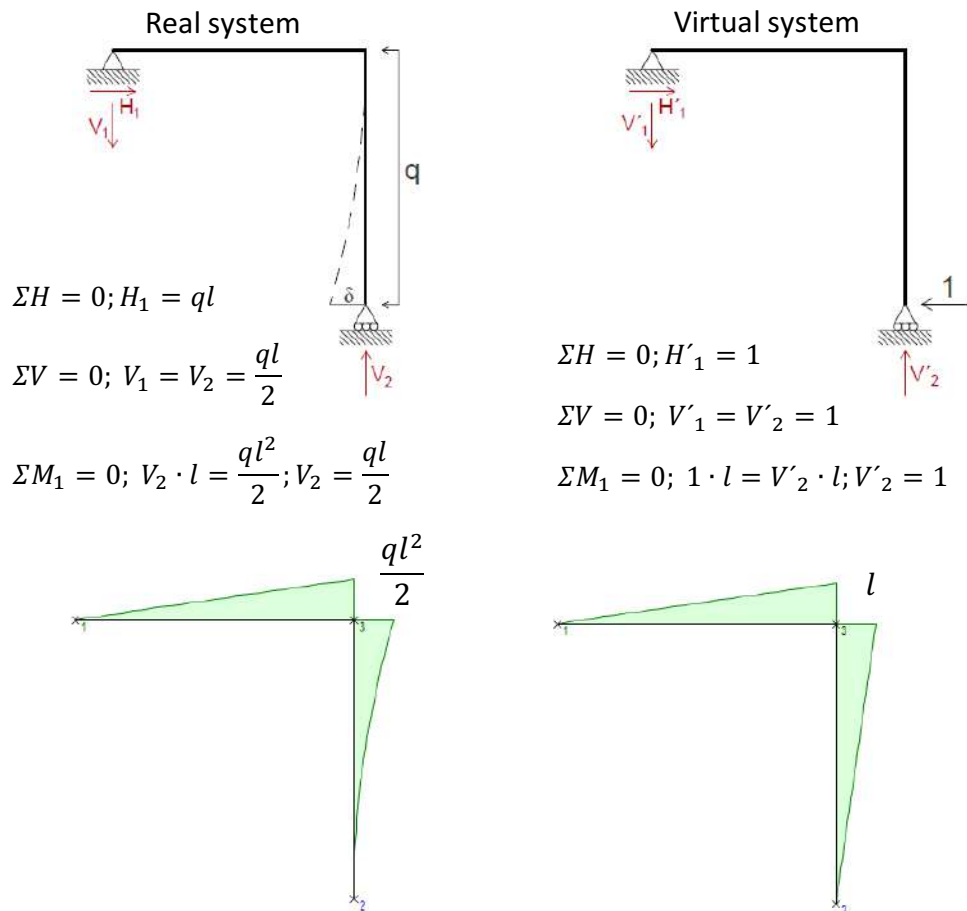


Data:

$$q = 2000 \text{ kg/m}$$

$$L = 4\text{m}$$

$$EI = 6,5 \times 10^{10} \text{ kgxcm}^2$$



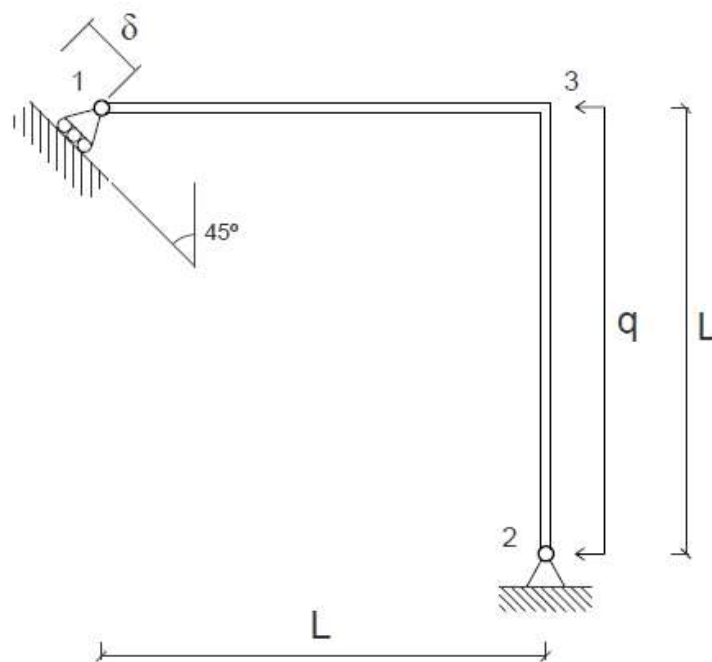
$$1' \cdot \delta = \int_0^l \frac{M \cdot M'}{EI} dx = \frac{1}{2EI} \cdot \frac{ql^4}{6} + \frac{1}{EI} \cdot \frac{ql^4}{8} = \frac{ql^4}{24EI} (2 + 3) = \frac{5ql^4}{24EI}$$

U / R	
	$\int_0^l MM' dx = \frac{1}{3} \cdot l \cdot \frac{ql^2}{2} \cdot l + \frac{1}{4} \cdot l \cdot \frac{ql^2}{2} \cdot l$

$$\int_0^l MM' dx = \frac{ql^4}{6} + \frac{ql^4}{8}$$

$$\delta = \frac{5ql^4}{24EI} = \frac{5 \cdot 20 \cdot 400^4}{24 \cdot 6,5 \cdot 10^{10}} = 1,64 \text{ cm}$$

EXERCISE 2. Using the unit load method, calculate the indicated displacement of the frame.

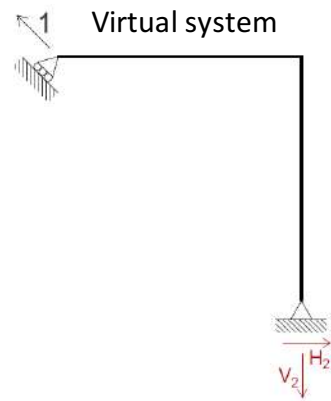
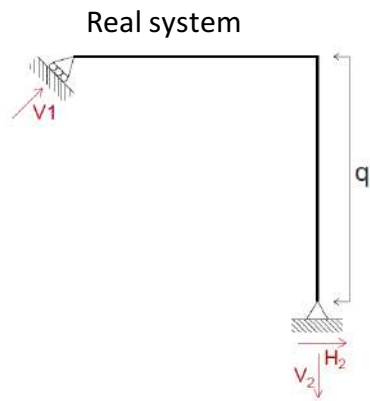


Data:

$$q = 2000 \text{ kg/m}$$

$$L = 4\text{m}$$

$$EI = 6,5 \times 10^{10} \text{ kgxcm}^2$$



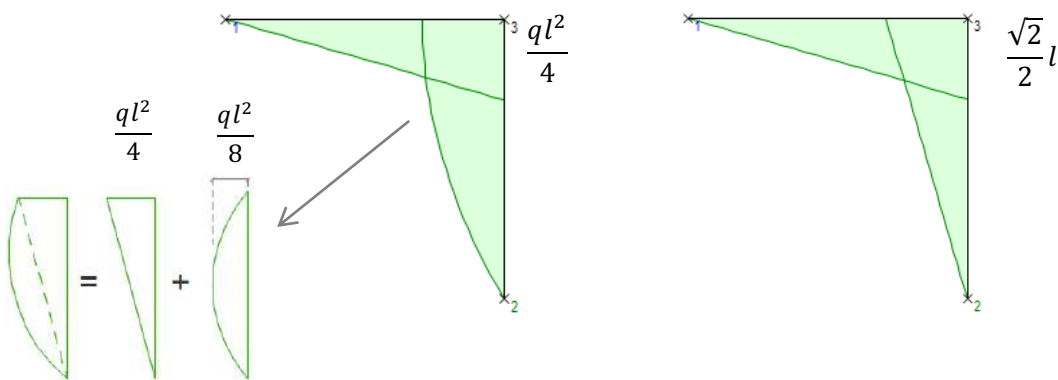
$$\Sigma H = 0; H_2 + V_1 \cdot \sin \alpha = ql; H_2 = ql - \frac{ql}{4} = \frac{3ql}{4}$$

$$\Sigma H = 0; H_2 = 1 \cdot \sin \alpha = \frac{\sqrt{2}}{2}$$

$$\Sigma V = 0; V_2 = V_1 \cos \alpha; V_2 = \frac{ql}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{2} = \frac{ql}{4}$$

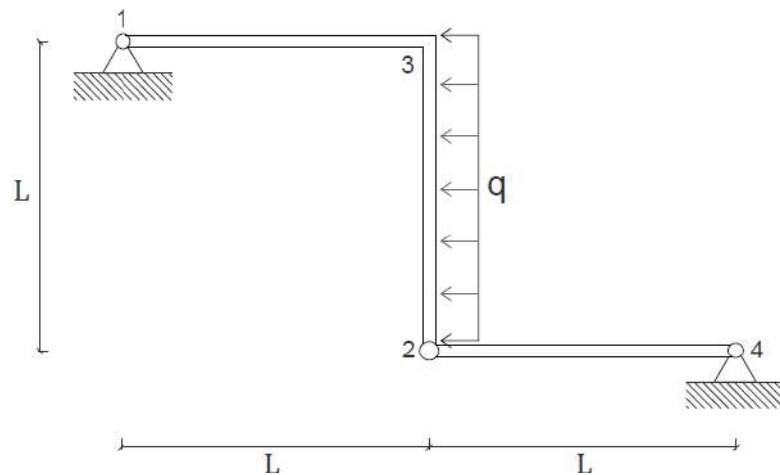
$$\Sigma V = 0; V_2 = 1 \cdot \cos \alpha = \frac{\sqrt{2}}{2}$$

$$\Sigma M_2 = 0; \frac{ql^2}{2} = V_1 \cdot l \cdot \sqrt{2}; V_1 = \frac{ql}{2\sqrt{2}}$$



$$1 \cdot \delta = \int_0^l \frac{M \cdot M'}{EI} dx = \frac{5 \cdot \sqrt{2}}{48} \cdot \frac{ql^4}{EI} = \frac{5 \cdot \sqrt{2} \cdot 20 \cdot 400^4}{48 \cdot 6,5 \cdot 10^{10}} = \frac{7,54 \cdot 10^{10}}{6,5 \cdot 10^{10}} = 1,16 \text{ cm}$$

EXERCISE 3



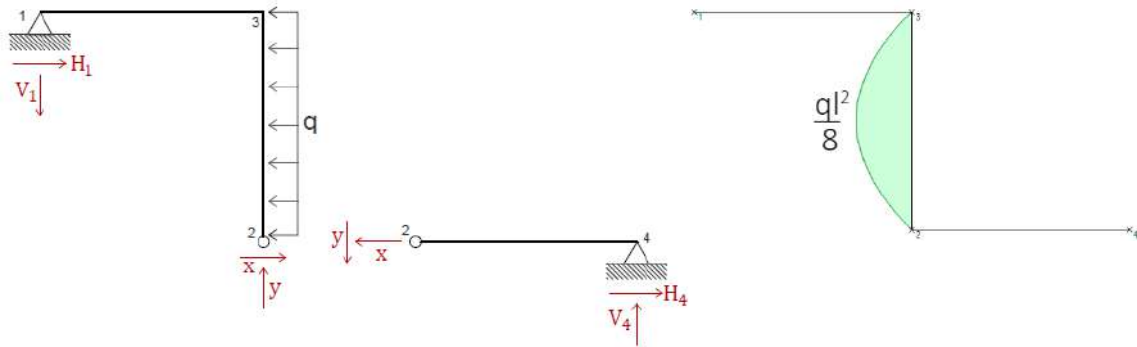
- Using the unit load method, calculate the value of the q distributed weight, so that the displacement of the 3rd node is 1 cm.
- With this value of q , calculate the rotation of the 3rd joint.
- Calculate the diagrams and the deformation with the q distributed load.

Data:

$$l = 4 \text{ m}$$

$$EI = \text{Constant} = 3,2 \cdot 10^{10} \text{ kg} \cdot \text{cm}^2$$

Real system



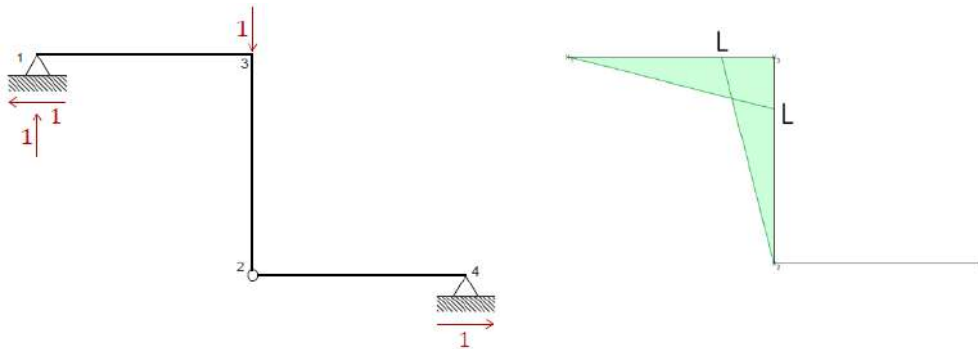
$$V_1 = y = 0$$

$$V_4 = y = 0$$

$$H_1 + x = q \cdot l; H_1 = \frac{ql}{2} \quad H_4 = x$$

$$V_1 \cdot l + \frac{ql^2}{2} - H_1 \cdot l = 0$$

Virtual system

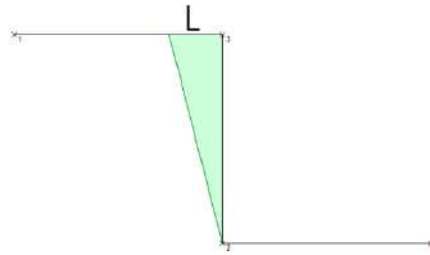
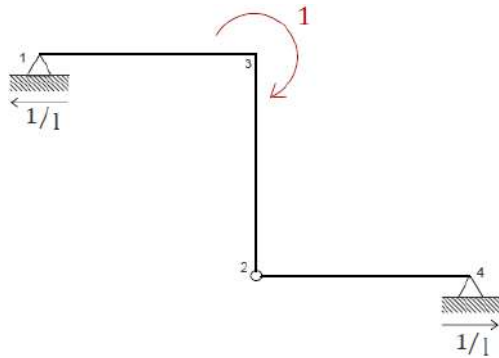


R U	
	$\delta_3 = \frac{1}{EI} \left(\frac{1}{3} \cdot M \cdot M' \cdot l \right) = \frac{ql^4}{24EI}$

$$\delta_3 = \frac{ql^4}{24EI} \rightarrow 1 = \frac{q \cdot 400^4}{24 \cdot 3,2 \cdot 10^{10}}$$

$$q = \frac{24 \cdot 3,2 \cdot 10^{10}}{400^4} = 30 \text{ kg/cm} = 3000 \text{ kg/m}$$

2. Node rotation



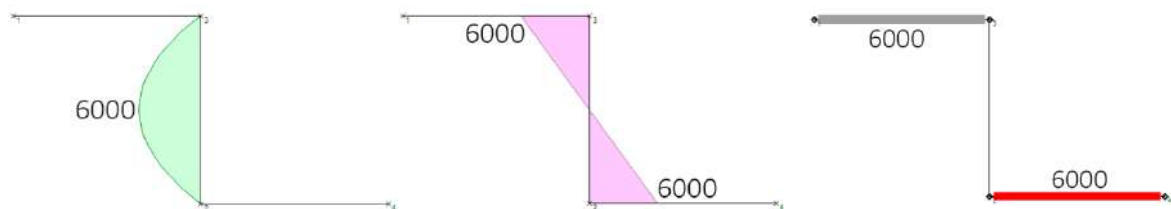
R	
U	

$$\sigma_3 = \frac{1}{EI} \left(\frac{1}{3} \cdot M \cdot M' \cdot l \right) = \frac{ql^3}{24EI}$$

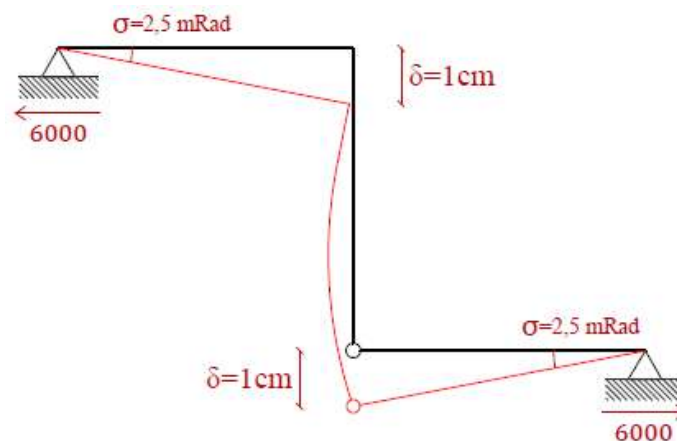
$$\sigma_3 = \frac{ql^3}{24EI} = \frac{30 \cdot 400^3}{24 \cdot 3,2 \cdot 10^6}$$

$$\sigma_3 = 2,5 \cdot 10^{-3} \text{ rad}; \quad \boxed{\sigma_3 = 2,5 \text{ mRad}}$$

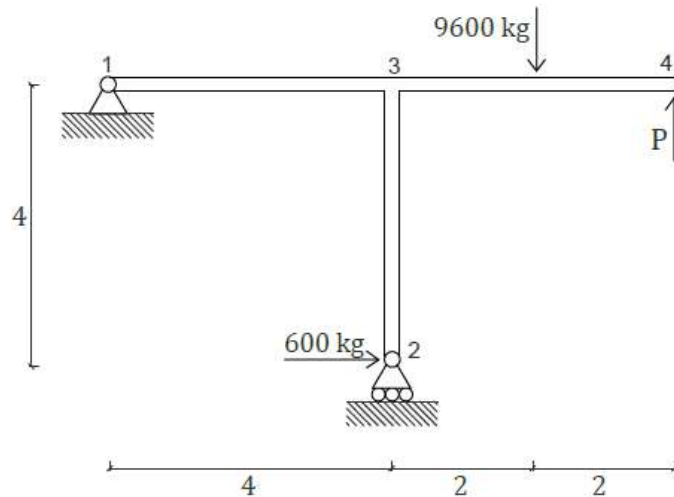
Diagrams



Reactions and deformation



EXERCISE 4

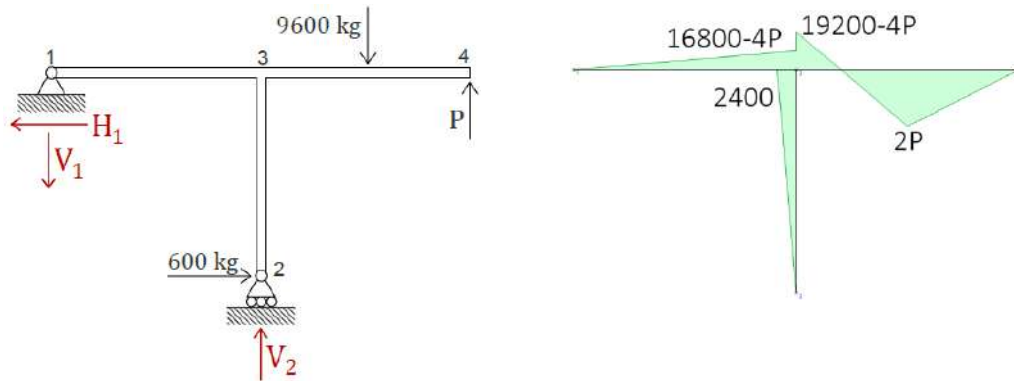


- Using the unit load method, calculate what has to be the value of P load, so that there is no displacement on the 4. node.
- With the P load, calculate the rotation on the 3. node and the displacement on the 2. support.
- Calculate the diagrams and the deformation with P load.

Data

$$EI = kte = 3,2 \cdot 10^{10} \text{ kg} \cdot \text{cm}^2$$

Real system

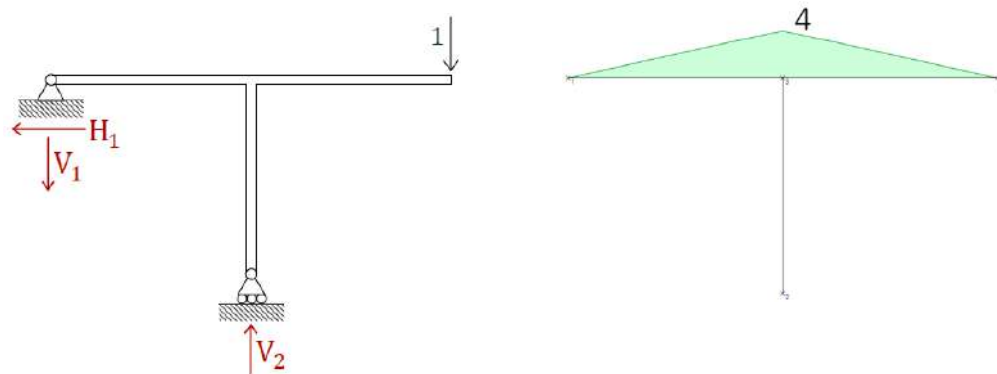


$$\Sigma H = 0 ; \boxed{H_1 = 600}$$

$$\Sigma V = 0 ; P + V_2 - V_1 - 9600 = 0 ; V_1 = -9600 + V_2 + P ; \boxed{V_1 = 4200 - P}$$

$$\Sigma M_1 = 0 ; 4 V_2 + 8P + 4 \cdot 600 - 9600 \cdot 6 = 0 ; V_2 = 14400 - 600 - 2P ; \boxed{V_2 = 13800 - 2P}$$

Virtual system



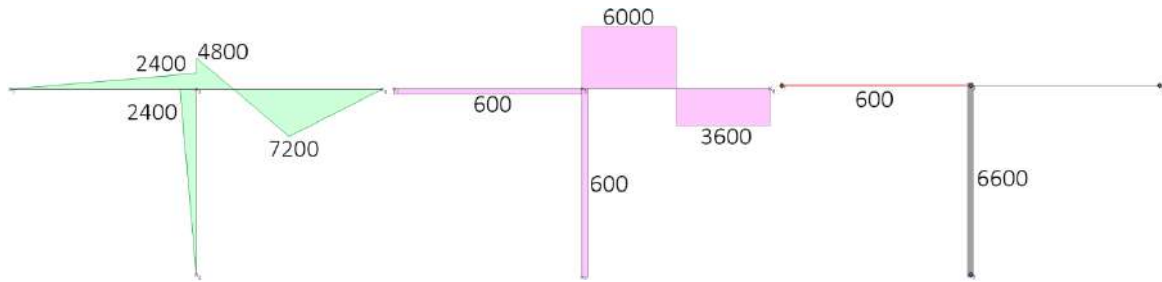
$$\Sigma H = 0 ; \boxed{H_1 = 0}$$

$$\Sigma V = 0 ; V_2 - V_1 - 1 = 0 \rightarrow V_1 = 2 - 1 ; \boxed{V_1 = 1}$$

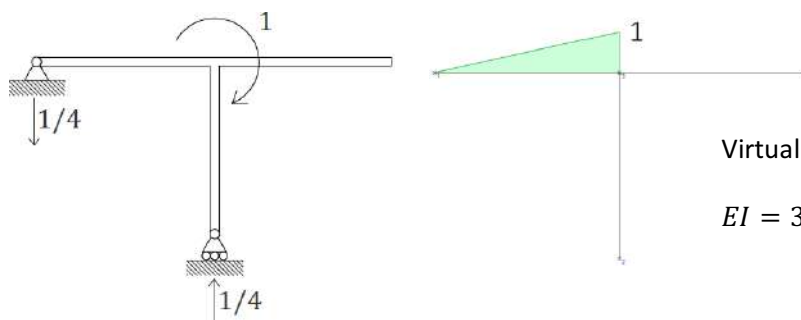
$$\Sigma M_1 = 0 ; -1 \cdot 8 + 4 \cdot V_2 ; \boxed{V_2 = 2}$$

4. node displacement

Diagrams with P load

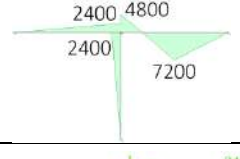


3. node rotation



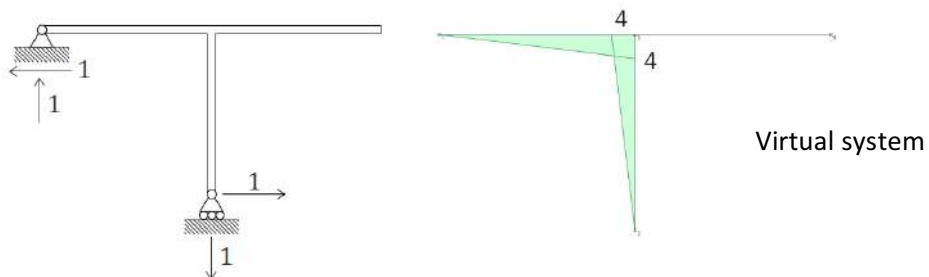
Virtual system:

$$EI = 3,2 \cdot 10^{10} \text{ kg} \cdot \text{cm}^2 \\ = 3,2 \cdot 10^6 \text{ kg} \cdot \text{m}^2$$

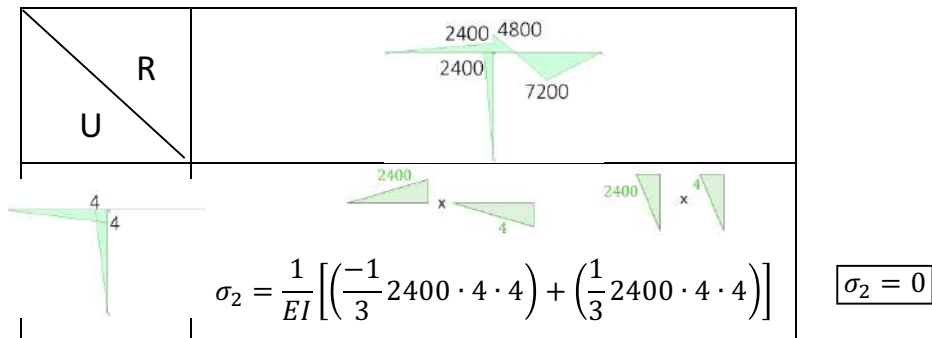
R U	
	$\sigma_3 = \frac{1}{EI} \left(\frac{1}{3} \cdot 1 \cdot 2400 \cdot 4 \right)$

$$\sigma_3 = \frac{3200}{3,2 \cdot 10^6} = 10^{-3} \text{ Rad} ; \boxed{\sigma_3 = 1 \text{ mRad}}$$

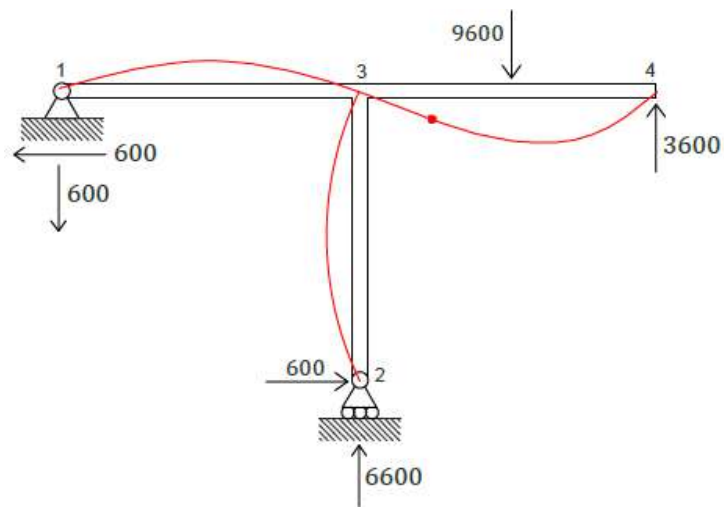
2. roller support displacement



Virtual system



Reactions and deformation



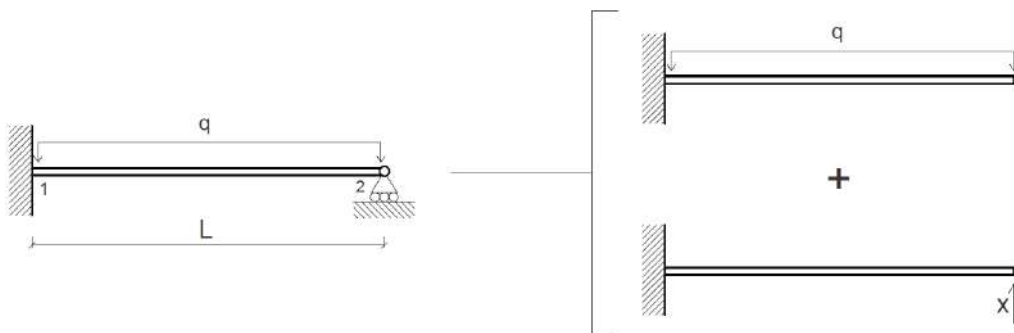
$$\Sigma H = 0 ; 600 = 600 \checkmark$$

$$\Sigma V = 0 ; 6600 + 3600 = 600 + 9600 ; 10200 = 10200 \checkmark$$

$$\Sigma M_1 = 0 ; 6600 \cdot 4 + 600 \cdot 4 + 3600 \cdot 8 = 9600 \cdot 6 ; 57600 = 57600 \checkmark$$

4.3.6. Unit load method. Hyperstatic frames.

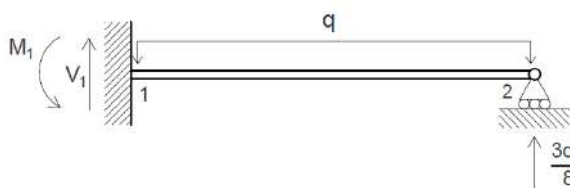
The unit load method can also be used to solve hyperstatic frames. We will have to follow this procedure: we transform as many unknown hyperstatic values (reactions, for example) as needed, into action forces to make the frame become isostatic. This way, we turn our hyperstatic structure into an isostatic structure with unknown actions. Finally, to find out the value of these actions we introduce deformation conditions. We will understand this procedure better with the following example.



We know that the vertical deflection on the 2. support is zero. Taking into account this condition, we can define the value of "x" using the unit load method.

$\begin{matrix} \text{R} \\ \diagdown \\ \text{U} \end{matrix}$	
	$\delta = \left(\frac{1}{4} M_1 \cdot M_2 \cdot l - \frac{1}{3} M_1 \cdot M_2 \cdot l \right) \frac{1}{EI}$

$$\delta = \frac{1}{EI} \left(\frac{1}{4} \cdot l \cdot \frac{ql^2}{2} \cdot l - \frac{1}{3} \cdot l \cdot xl \cdot l \right) = 0; \frac{ql^4}{8} - \frac{x \cdot l^3}{3} = 0; \frac{x \cdot l^3}{3} = \frac{ql^4}{32}; x = \frac{3ql}{8}$$

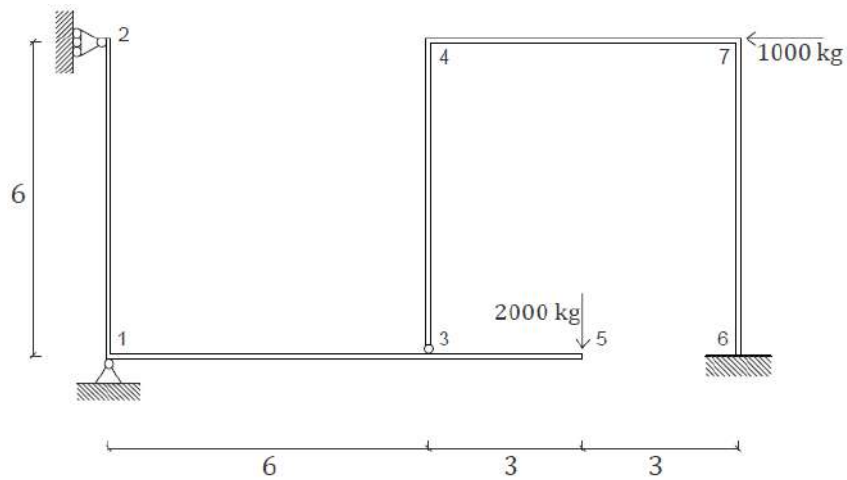


$$\Sigma V = 0; ql = V_1 + \frac{3ql}{8}; V_1 = \frac{5ql}{8}$$

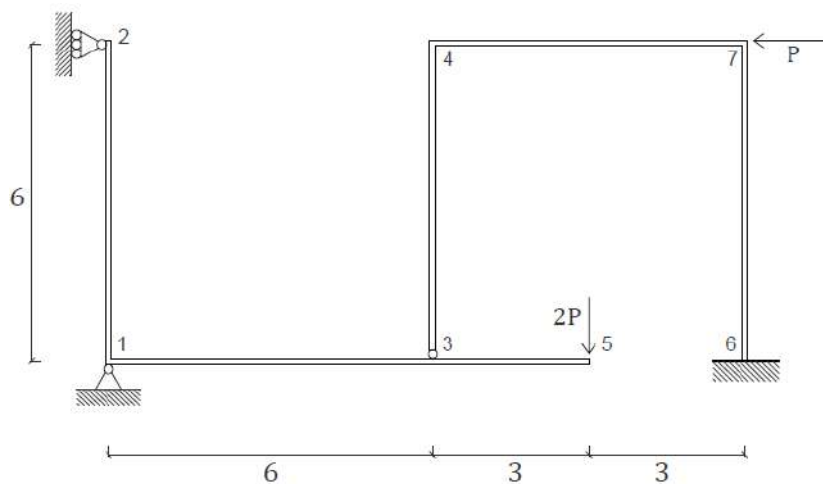
$$\Sigma M_1 = 0; \frac{ql^2}{2} = M_1 + \frac{3ql}{8} \cdot l; M_1 = \frac{ql^2}{8}$$

4.3.7. Unit load method. Exercises with hyperstatic frames

EXERCISE 1



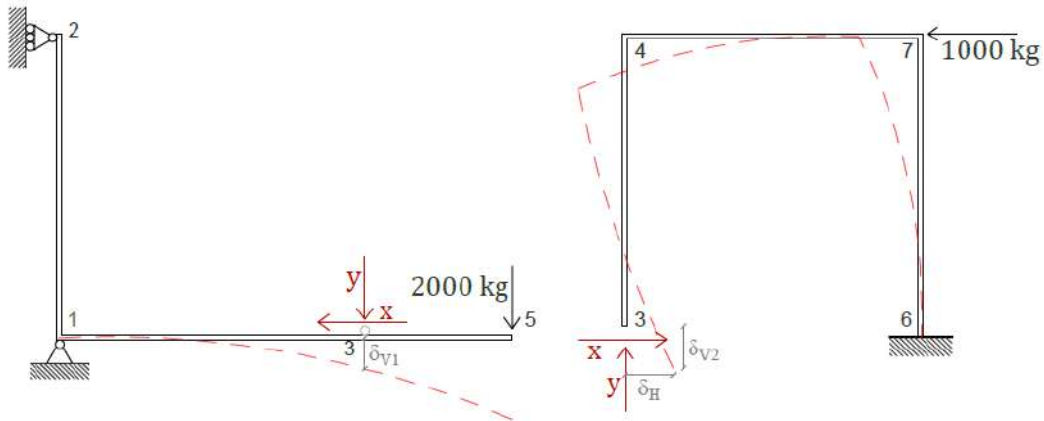
- Using the unit load method define the bending moment, shear and axial diagrams. Calculate the reactions and draw the deformation.
- Analyse in the same structure the following load situation. What has to be the value of P load for the vertical displacement of the 3. node to be 3 cm



Data:

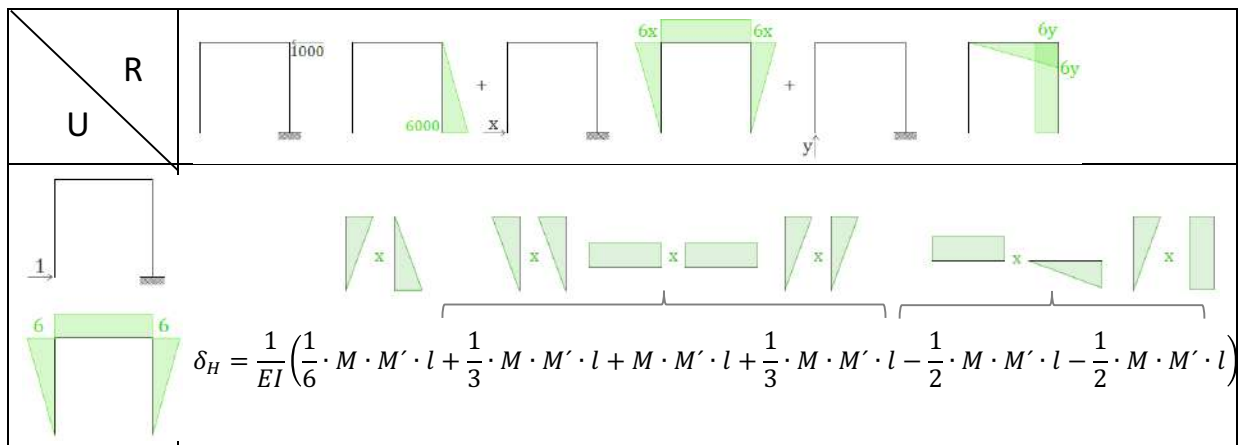
$$EI = \text{Constant} = 5,284 \cdot 10^6 \text{ kg} \cdot \text{cm}^2$$

We can split the structure in two different parts:



Deformation compatibility $\delta_H = 0$; $\delta_{V1} = \delta_{V2}$

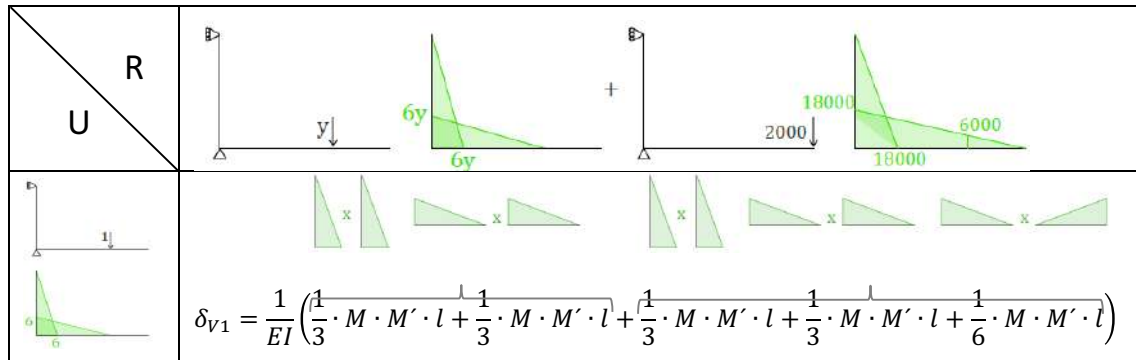
1) $\delta_H = 0$



$$\delta_H = \frac{1}{EI} \left(\frac{1}{6} \cdot 6000 \cdot 6 \cdot 6 + \frac{1}{3} \cdot 6 \cdot 6x \cdot 6 + 6 \cdot 6x \cdot 6 + \frac{1}{3} \cdot 6 \cdot 6x \cdot 6 - \frac{1}{2} \cdot 6 \cdot 6y \cdot 6 - \frac{1}{2} \cdot 6 \cdot 6y \cdot 6 \right)$$

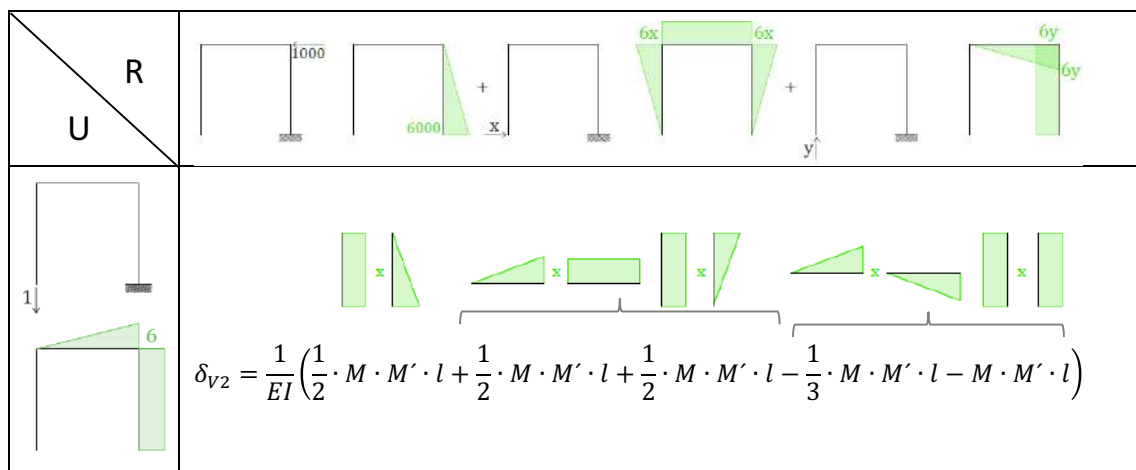
$$\delta_H = \frac{1}{EI} (36000 + 360x - 216y) = 0 ; \boxed{1000 + 10x - 6y = 0}$$

2) $\delta_{V1} = \delta_{V2}$



$$\delta_{V1} = \frac{1}{EI} \left(\frac{1}{3} \cdot 6 \cdot 6y \cdot 6 + \frac{1}{3} \cdot 6 \cdot 6y \cdot 6 + \frac{1}{3} \cdot 6 \cdot 18000 \cdot 6 + \frac{1}{3} \cdot 6 \cdot 18000 \cdot 6 + \frac{1}{6} \cdot 6 \cdot 6000 \cdot 6 \right)$$

$$\delta_{V1} = \frac{1}{EI} (144y + 468000)$$



$$\delta_{V2} = \frac{1}{EI} \left(\frac{1}{2} \cdot 6 \cdot 6000 \cdot 6 + \frac{1}{2} \cdot 6 \cdot 6x \cdot 6 + \frac{1}{2} \cdot 6 \cdot 6x \cdot 6 - \frac{1}{3} \cdot 6 \cdot 6y \cdot 6 - 6 \cdot 6y \cdot 6 \right)$$

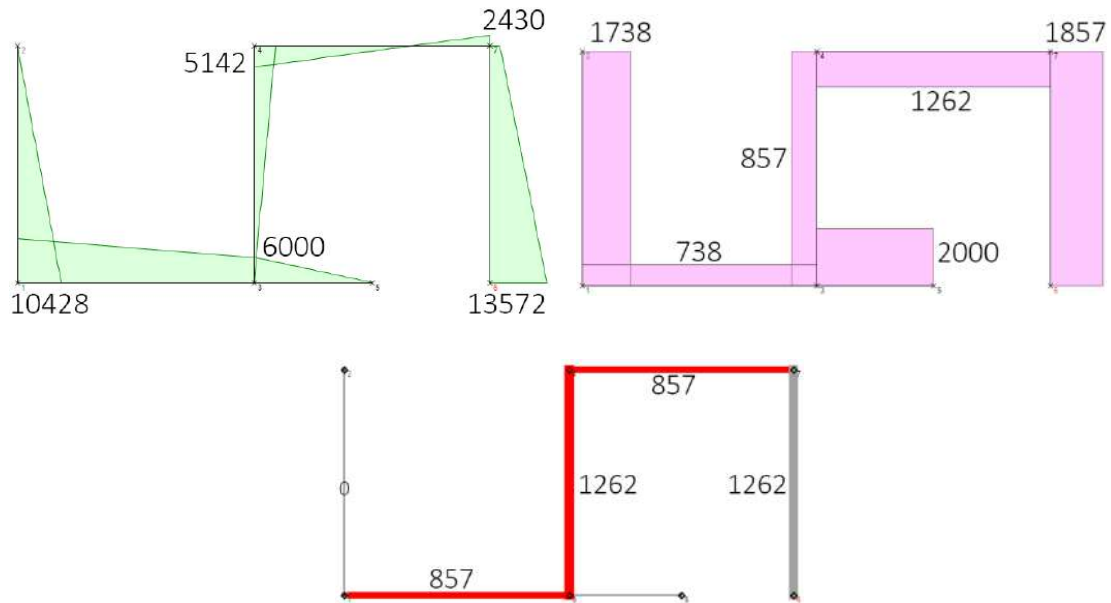
$$\delta_{V2} = \frac{1}{EI} (10800 + 216x - 288y)$$

$$\delta_{V1} = \delta_{V2}; \frac{1}{EI} (144y + 468000) = \frac{1}{EI} (10800 + 216x - 288y)$$

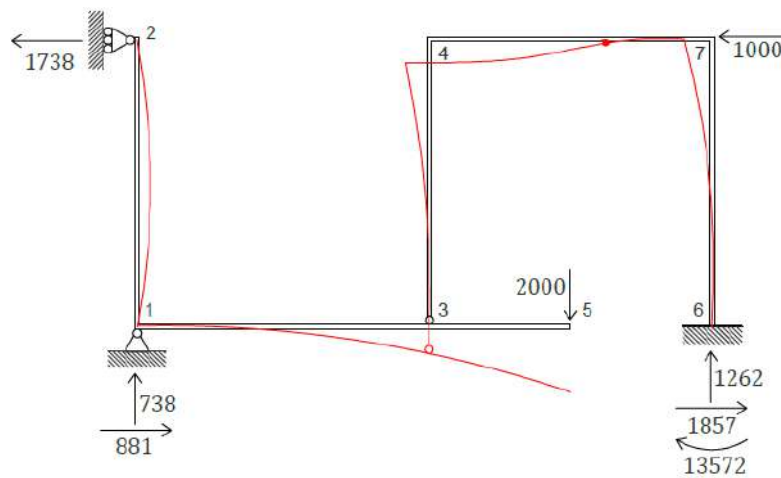
$$360000 - 216x + 432y = 0; \boxed{10000 - 6x + 12y = 0}$$

$$\left. \begin{array}{l} 1000 + 10x - 6y = 0 \\ 10000 - 6x + 12y = 0 \end{array} \right\} x = -857 \text{ kg}; y = -1262 \text{ kg}$$

Diagrams:



Reactions and deformation



$$\Sigma H = 0 ; 1000 + 1738 = 881 + 1857 ; 2738 = 2738 \checkmark$$

$$\Sigma V = 0 ; 2000 = 1262 + 738 ; 2000 = 2000 \checkmark$$

$$\begin{aligned} \Sigma M_1 = 0 ; & (1738 + 1000) \cdot 6 + 1262 \cdot 12 - 2000 \cdot 9 - 13572 \\ & = 0 ; 16428 + 15144 - 18000 - 13572 = 0 ; 0 = 0 \checkmark \end{aligned}$$

P load value

$$\delta_{V1} = \frac{1}{EI} (144y + 468000) = \frac{286272}{EI} = \frac{286272}{5,284 \cdot 10^6} = 0,054 \text{ m} = 5,4 \text{ cm}$$

Then, using a simple mathematic rule:

$$\left. \begin{array}{l} P = 1000 \rightarrow \delta_{V1} = 5,4 \text{ cm} \\ P = ? \rightarrow \delta_{V1} = 3 \text{ cm} \end{array} \right\} \boxed{P = \frac{3 \cdot 1000}{5,4} = 555,5 \text{ kg}}$$

5. RESOLUTION OF HYPERSTATIC FRAMES BY THE CROSS METHOD

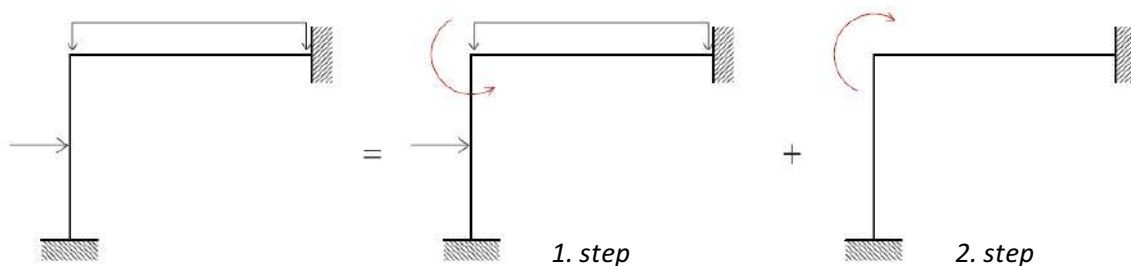
The Cross method name comes from the Hardy Cross American engineer, the man who developed and published the method for the first time in 1930. Thanks to this method, we can calculate structures with a high degree of hyperstaticity, just by doing simple and repetitive operations. On the way, we consider the axial and shear strengths negligible when calculating the deformation of the bars, and we focus on the bending moment strength. This simplification will be particularly suitable for frames with slender bars, which are going to be working mainly under flection (most of the construction structures are this type).

These features allow simplifying the hand calculations for concrete frames. Because of this, it became the main calculation method since its publication and until the arrival of the computer calculation software.

5.1. NON SWAY FRAMES

5.1.1. Introduction, methodology and previous concepts

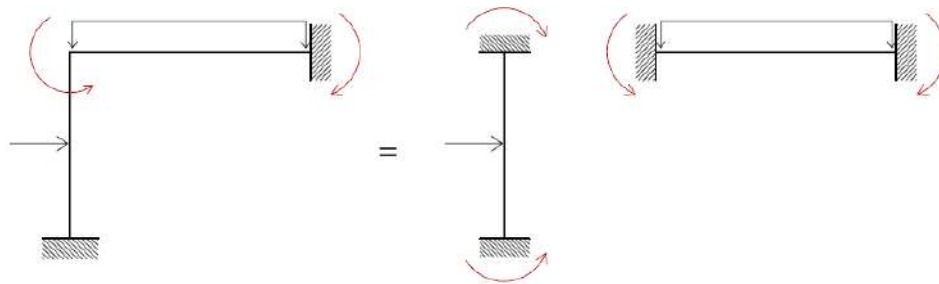
Let us examine a simple frame composed of two bars joined by a stiff node. Taking into account the principle of superposition, the addition of these two situations will be the equivalent of our structure.



1. step

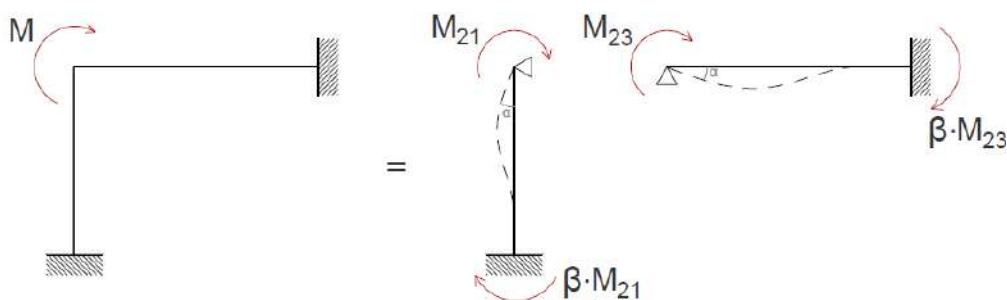
We apply a moment to the stiff joint to restrain the rotation. This allows us to separate the bars, representing the ends leading to the node as fixed ends. This way, this first state can be solved by the Fixed End Moment (section 5.1.2. of these notes).

It means, by low hyperstaticity simple bars, that we will get solved by using a table (Appendix 2 of these notes).



2. step

In this second step we always have a frame with no more actions than moments in the nodes. Therefore, we will solve this second state by using the distribution and transmission factors (sections 5.1.3 and 5.1.4 of these notes)

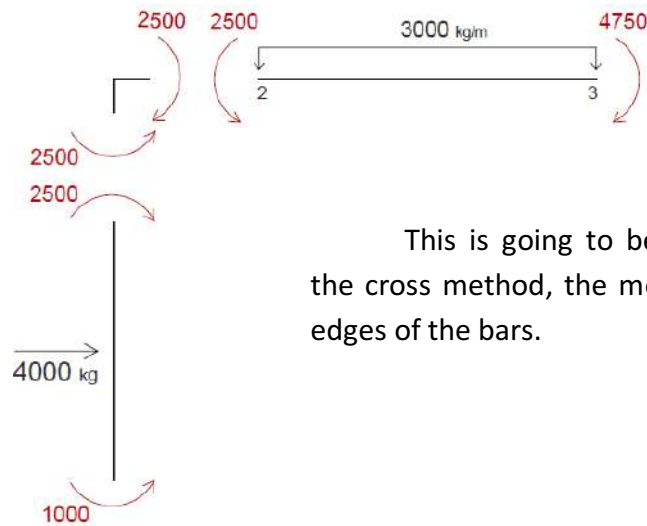
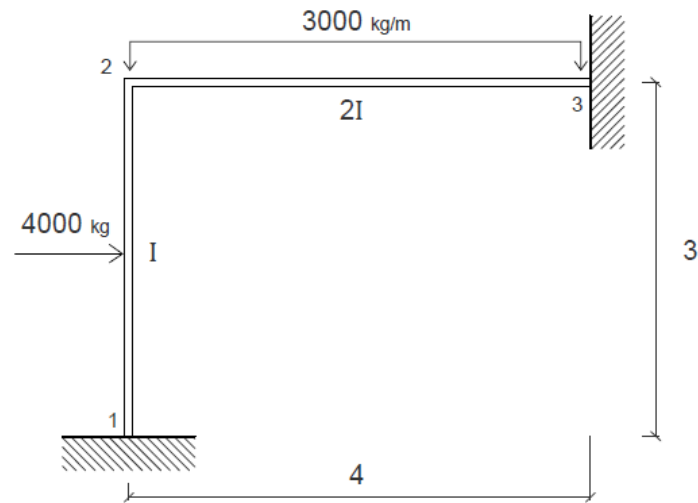


Methodology:

Summarizing as much as possible the Cross method, we could say that it is the application of the principle of superposition in a specific way, in order to be able to solve separately each bar of the frame, through simple and repetitive calculations.

As a result of the previous operation, we will get the bending moment in the end of each bar. Once we know these moments, we will solve the frame in a simple way, as we can see in the next example.

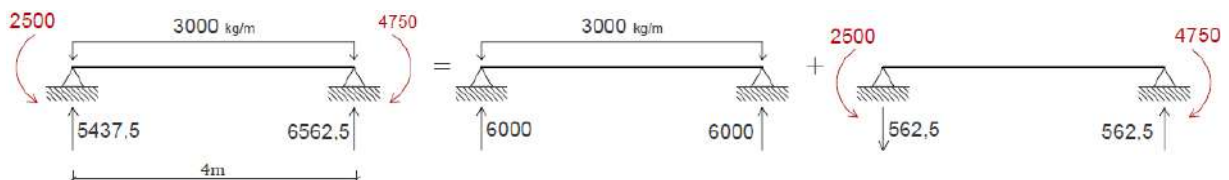
Example



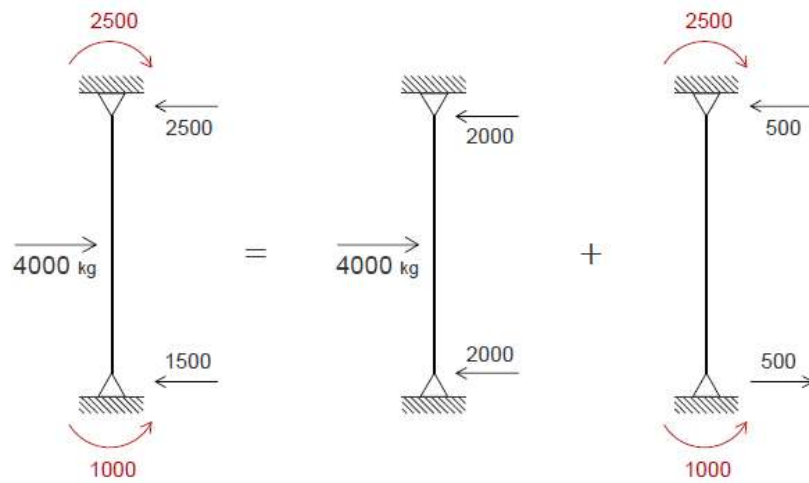
This is going to be the result of the cross method, the moments on the edges of the bars.

Once we know the moments on these ends, it's easy to calculate the shear strength along the bars:

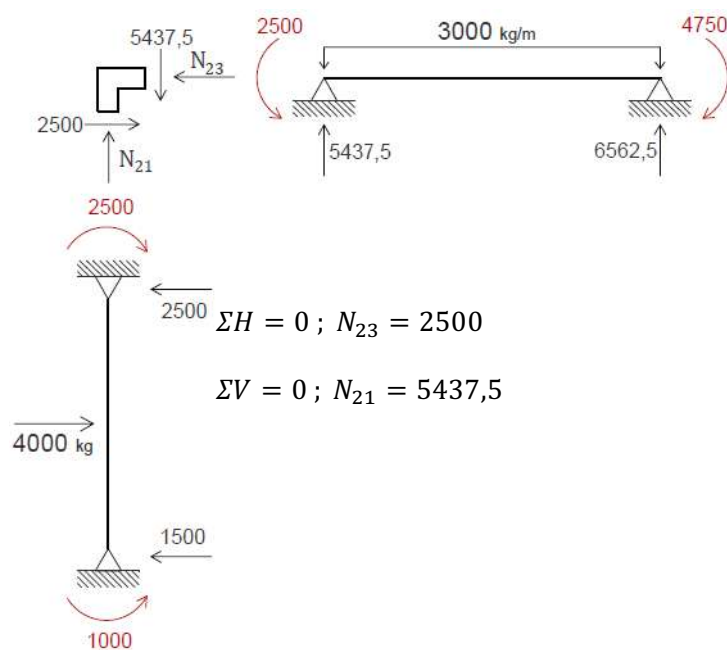
2-3 bar



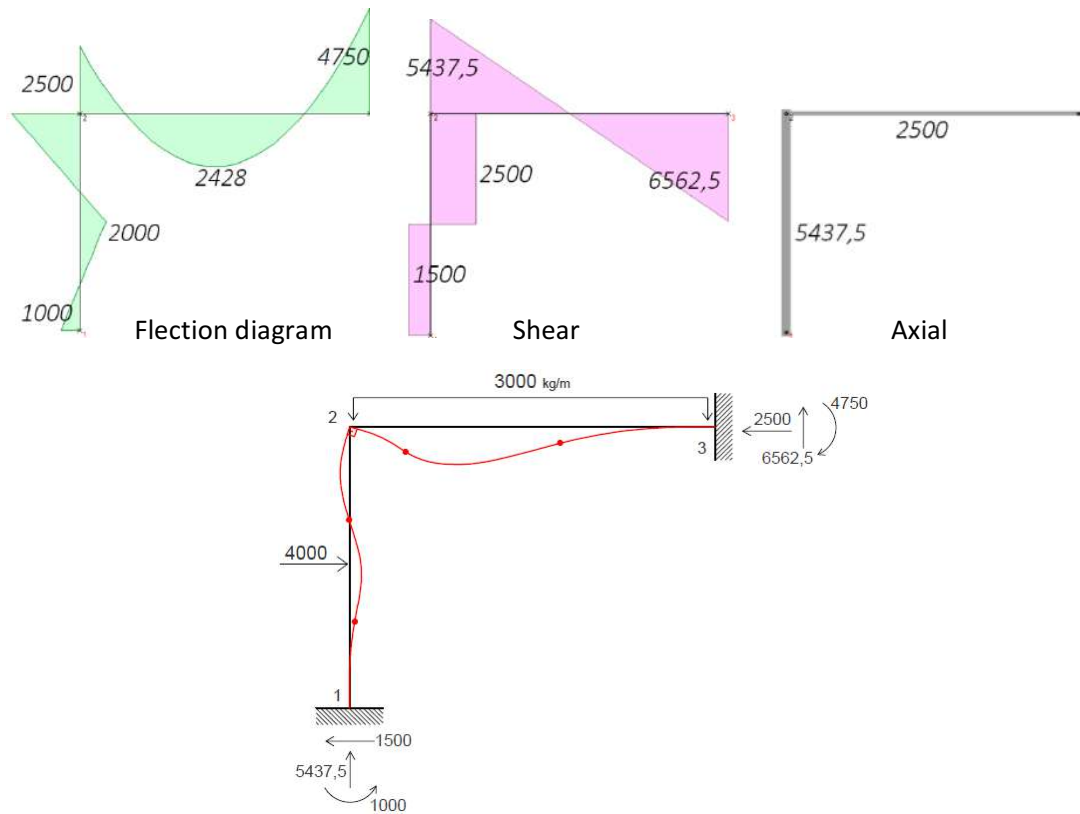
1-2 bar



After obtaining the shear strengths, by applying the equilibrium equations in the nodes, we will get the axial strengths:



Following all the previous steps and once we have all the forces, we will get the diagrams, reactions and deformation of our structure.



The previous methodology can be applicable in any structure:

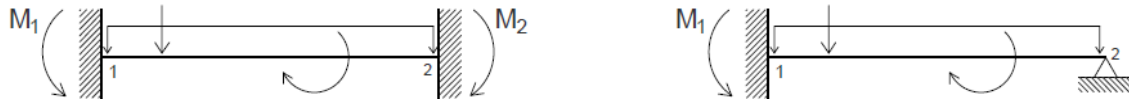
- 1) Bar's ends moments calculation by the cross method
- 2) Independent bar analysis
- 3) Calculation of the shear strengths by equilibrium in the bars
- 4) Calculation of the axial strengths by equilibrium in the nodes

So now we know the purpose of the method: obtain the moments in the ends of the bars. But before beginning with the Cross method itself, we will need to study 4 more basic concepts that are necessary for its development:

- 1) Fixed end moments
- 2) Stiffness
- 3) Distribution factor
- 4) Carry over factor

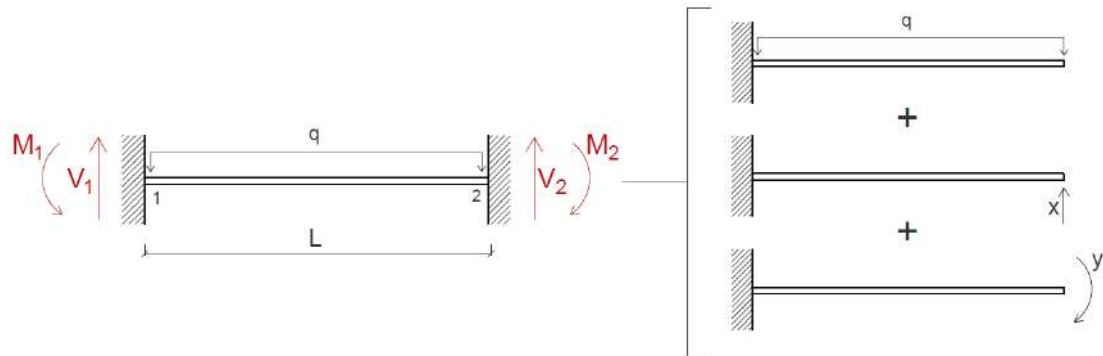
5.1.2.Fixed end moments (FEM)

In the context of this method, the FEM are the moments on the fixed ends of simple bars, whatever the loading they support is.



Hence, all the structures are going to be simple, just with one bar. They will be low level hyperstatic structures, and we can solve them easily by different methods.

We will solve the following example by using the unit load method:



In this case, we know:

$$\delta_2 = 0; \vartheta_2 = 0$$

U R	$\frac{-ql^2}{2}$ $x \cdot l$ $-y$
	$\times$ $\times$ $\times$ $\delta = \frac{1}{EI} \left(\frac{1}{4} M_1 \cdot M_2 \cdot l - \frac{1}{3} M_1 \cdot M_2 \cdot l + \frac{1}{2} M_1 \cdot M_2 \cdot l \right)$
	$\times$ $\times$ $\times$ $\vartheta = \frac{1}{EI} \left(\frac{1}{3} M_1 \cdot M_2 \cdot l - \frac{1}{2} M_1 \cdot M_2 \cdot l + M_1 \cdot M_2 \cdot l \right)$

$$1) \delta = \frac{1}{EI} \left(\frac{1}{4} \frac{ql^2}{2} l \cdot l - \frac{1}{3} x \cdot l \cdot l \cdot l + \frac{1}{2} y \cdot l \cdot l \right) = 0; \frac{ql^4}{8} - \frac{x \cdot l^3}{3} + y \frac{l^2}{2} = 0; \frac{ql^2}{8} - \frac{x \cdot l}{3} + \frac{y}{2} = 0$$

$$2) \vartheta = \frac{1}{EI} \left(\frac{1}{3} \frac{ql^2}{2} 1 \cdot l - \frac{1}{2} x \cdot l \cdot 1 \cdot l + y \cdot 1 \cdot l \right) = 0; \frac{ql^3}{6} - \frac{x \cdot l^2}{2} + y l = 0; \frac{ql^2}{6} - \frac{x \cdot l}{2} + y = 0$$

$$1) y = 2 \left(\frac{x \cdot l}{3} - \frac{ql^2}{8} \right) \quad \left\{ \begin{array}{l} 2 \left(\frac{x \cdot l}{3} - \frac{ql^2}{8} \right) = \frac{x \cdot l}{2} - \frac{ql^2}{6}; \frac{2x}{3} - \frac{x}{2} = \frac{ql}{4} - \frac{ql}{6}; \frac{x}{6} = \frac{ql}{12}; \boxed{x = \frac{ql}{2}} \end{array} \right.$$

$$2) y = \frac{x \cdot l}{2} - \frac{ql^2}{6} \quad \left\{ \begin{array}{l} y = \frac{x \cdot l}{2} - \frac{ql^2}{6}; y = \frac{ql^2}{4} - \frac{ql^2}{6}; \boxed{y = \frac{ql^2}{12}} \end{array} \right.$$

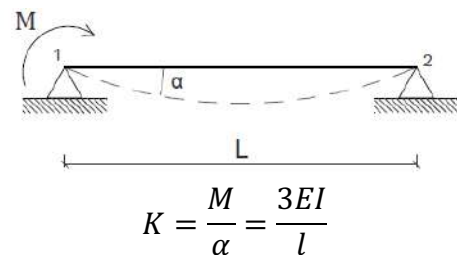
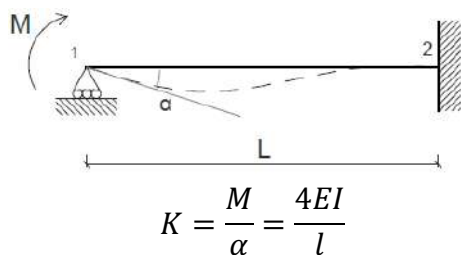
By symmetry: $M_1 = M_2 = \frac{ql^2}{12}; V_1 = V_2 = \frac{ql}{2}$

When doing the exercises, we will get the fixed end moment from a table, depending on the kind of load supported (Fixed End Moments Table. Appendix 2 of these notes).

5.1.3. Stiffness

The stiffness of a bar is the relationship between a moment we apply in a hinged end of a simple bar, and the rotation it causes. Stiffness is expressed with k letter.

In the following structures, we will solve the stiffness against the rotation, by using the unit load method:



Stiffness of a hinged-fixed bar

First we find out the reaction in the fixed end by applying the unit load method for hyperstatic structures. The deformation condition is:

$$\vartheta_2 = 0$$

	$\vartheta = \frac{1}{EI} \left(\frac{-1}{6} M_1 \cdot M_2 \cdot l + \frac{1}{3} M_1 \cdot M_2 \cdot l \right)$

$$\vartheta = \frac{1}{EI} \left(\frac{-1}{6} 1 \cdot M \cdot l + \frac{1}{3} 1 \cdot x \cdot l \right) = 0; \vartheta = \frac{1}{EI} \left(\frac{-Ml}{6} + \frac{x l}{3} \right) = 0; \frac{x l}{3} = \frac{Ml}{6}; x = \frac{M}{2}$$

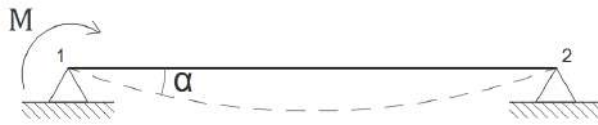
Now we can calculate α angle:

	$\alpha = \frac{1}{EI} \left(\frac{1}{3} M_1 \cdot M_2 \cdot l + \frac{-1}{6} M_1 \cdot M_2 \cdot l \right)$

$$\alpha = \frac{1}{EI} \left(\frac{1}{3} 1 \cdot M \cdot l - \frac{1}{6} 1 \cdot \frac{M}{2} \cdot l \right); \alpha = \frac{Ml}{EI} \left(\frac{1}{3} - \frac{1}{12} \right); \alpha = \frac{Ml}{4EI}$$

$$K = \frac{M}{\alpha} = \frac{M}{\frac{Ml}{4EI}} = \frac{4EI}{l}; \boxed{K = \frac{4EI}{l}}$$

Double hinged bar stiffness



Calculation of α angle:

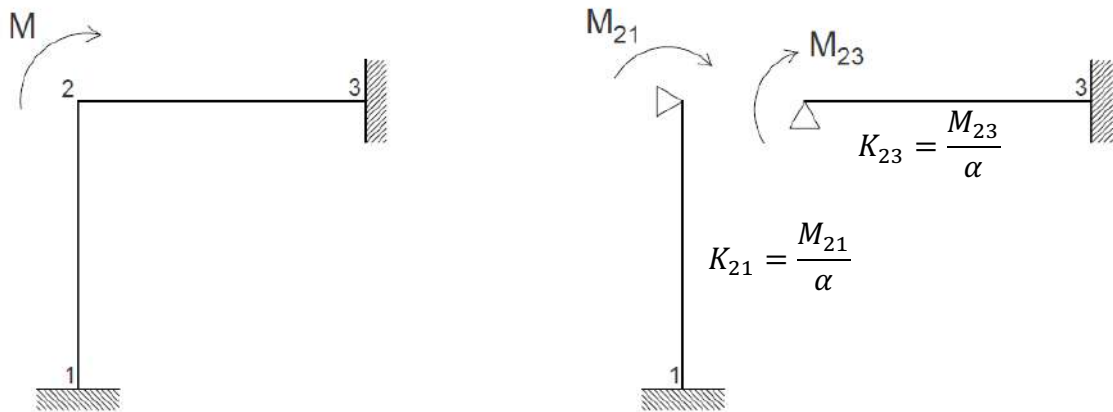
R U	
	$\alpha = \frac{1}{EI} \left(\frac{1}{3} M_1 \cdot M_2 \cdot l \right) = \frac{1}{EI} \left(\frac{1}{3} 1 \cdot M \cdot l \right) = \frac{Ml}{3EI}$

Then, stiffness value:

$$K = \frac{M}{\alpha} = \frac{M}{\frac{Ml}{3EI}} = \frac{3EI}{l}; \quad \boxed{K = \frac{3EI}{l}}$$

5.1.4. Distribution factor

Let's suppose that we have a discharged structure with an only stiff joint in which an "n" number of bars converge. And let's assume that the opposite ends of these bars are fixed-ends or hinged support. When applying an "M" moment on that stiff joint, the moment percentage that is transmitted to each bar is called *distribution factor*. Distribution factor is expressed with *r* letter.



$$K_{21} = \frac{M_{21}}{\alpha}; K_{23} = \frac{M_{23}}{\alpha}$$

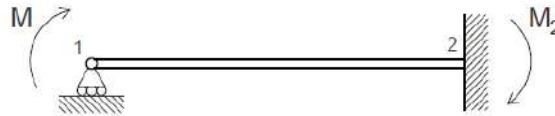
$$\alpha = \frac{M_{21}}{K_{21}} = \frac{M_{23}}{K_{23}} = \frac{M_{21} + M_{23}}{K_{21} + K_{23}}; \left(\frac{1}{2} = \frac{2}{3} = \frac{3}{6} = \frac{4}{8} = \frac{1+2+3+4}{2+4+6+8} = \frac{10}{20} \right)$$

$$\frac{M_{21}}{K_{21}} = \frac{\Sigma M_i}{\Sigma K_i} = \frac{M}{\Sigma K_i}; M_{21} = M \frac{K_{21}}{\Sigma K_i}; M_{23} = M \frac{K_{23}}{\Sigma K_i} \rightarrow r_{23}: \text{Distribution factor}$$

$$r_{21} + r + \dots r_{2n} = \frac{K_{21}}{\Sigma K_i} + \frac{K_{23}}{\Sigma K_i} + \dots \frac{K_{2n}}{\Sigma K_i} = \frac{\Sigma K_i}{\Sigma K_i} = 1 \text{ (percentage)}$$

5.1.5. Carry over factor

Let's assume that we have a fixed-hinged bar. When we apply an "M" moment on the hinged end, we cause a reaction in the fixed end that can be expressed as a percentage of this moment. That is what we call *carry over factor*. Carry over factor is expressed with β letter.



$$M_2 = \frac{M}{2} \text{ (calculated on the previous section)}$$

Carry over factor is the relation between the applied moment and the generated reaction:

$$\beta = \frac{M_2}{M} \rightarrow \beta = \frac{\frac{M}{2}}{M} = \frac{1}{2}; \beta = \frac{1}{2}$$

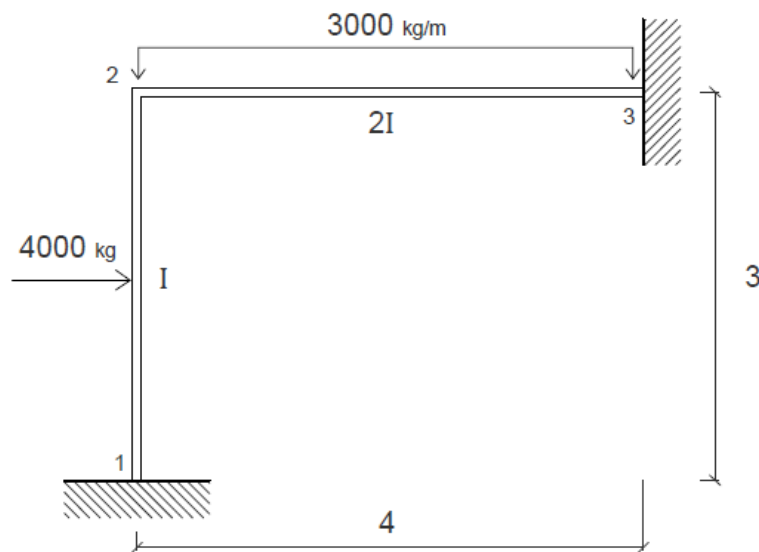
Through this factor, once we know the actions, we can directly calculate the reactions:

$$M_2 = \beta \cdot M = \frac{1}{2} \cdot M$$

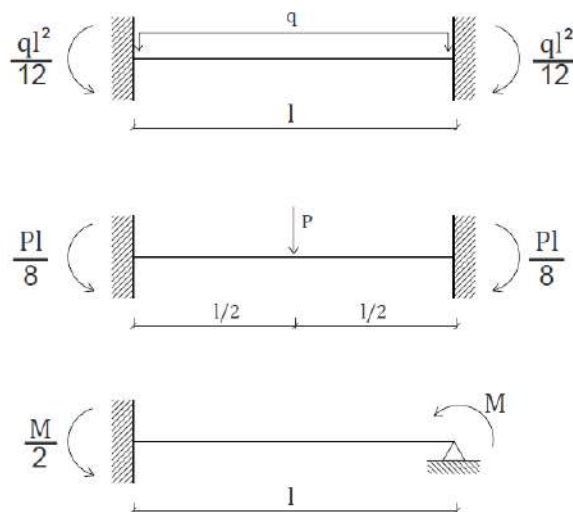
This four concepts (FEM; K; r; β) will be enough to solve any non-sway frame by using the Cross method.

5.1.6. Non sway orthogonal frames. One node frames.

In this section we will solve a frame by Using the Cross method. We will calculate the reactions, and draw bending moment, shear and axial diagrams. Also we will draw the deformation. Through this process, we will understand the full method, and will learn applying the concepts we have seen in the previous sections.



We will need the following data to solve the frame:



First of all, we need to calculate the FEM and the distribution factor to approach the Cross method table. It is the chart where we will calculate the moments in the ends of the bars.

Fixed end Moment (FEM)

$$M_{23} = M_{32} = \frac{ql^2}{12} = \frac{3000 \cdot 4^2}{12} = 4000 \text{ kgm}$$

$$M_{21} = M_{12} = \frac{Pl}{8} = \frac{4000 \cdot 3}{8} = 1500 \text{ kgm}$$

Stiffness

$$K_{23} = \frac{4EI_{23}}{l} = \frac{4E \cdot 2I}{4} = 2EI$$

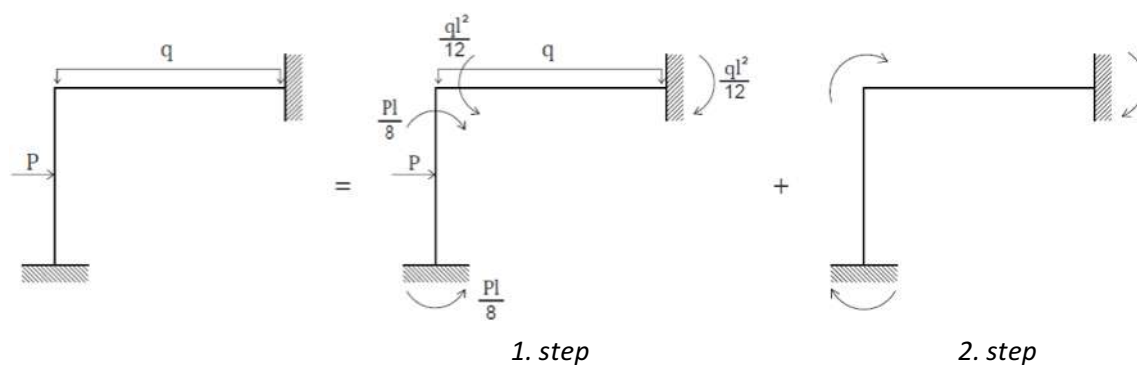
$$K_{21} = \frac{4EI_{21}}{l} = \frac{4E \cdot I}{3} = 1,3\hat{3}EI$$

Distribution factors in node 1

$$r_{23} = \frac{K_{23}}{K_{21} + K_{23}} = \frac{2EI}{(1,3\hat{3} + 2)EI} = 0,6$$

$$r_{21} = \frac{K_{21}}{K_{21} + K_{23}} = \frac{1,3\hat{3}EI}{(1,3\hat{3} + 2)EI} = 0,4$$

Diagram

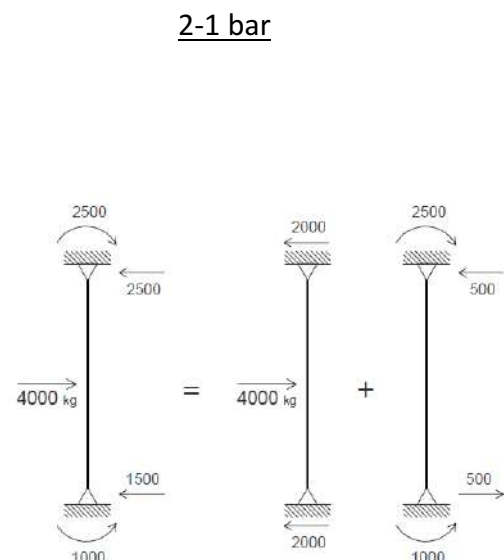
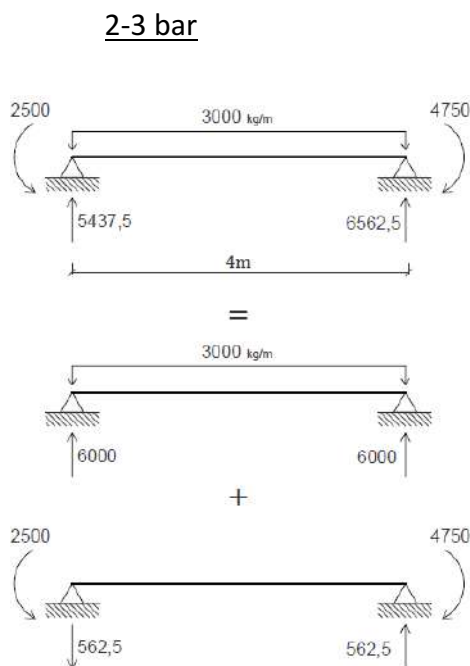


This table represents the moments existing in the ends of the bars during the two steps reflected in the diagram. Each column belongs to the ending of a bar. In stiff nodes we have an additional column for calculations. In this column we calculate the sum of moments that restrain the rotation. Then we apply the opposite moments and the carry over factors to release the nodes. In the last line of the table we have the sum of all moments applied in each bar's end during the two steps of the process. It is the result we are looking for to solve the frame.

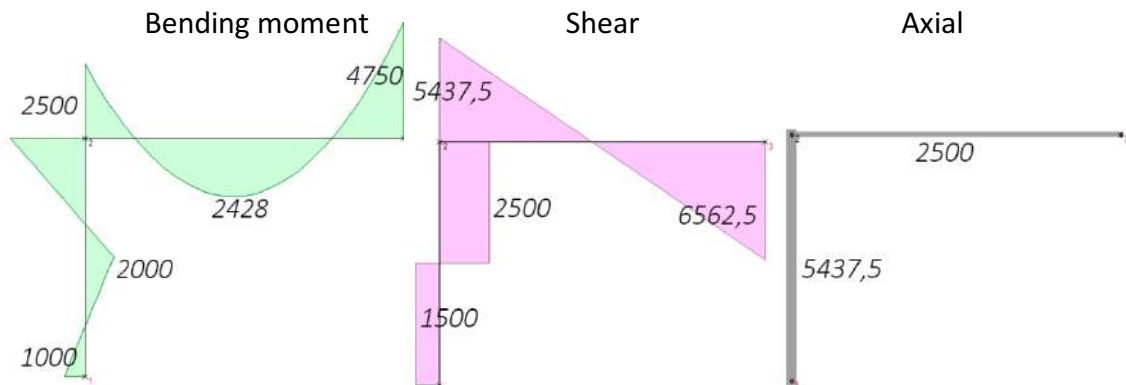
Cross Table

1	2			3	← Nodes / Supports
12		21	23	32	← Bar ends
-		0,4	0,6	-	← Distribution factors
1500	2500	-1500	4000	-4000	← FEM
-500	-2500	-1000	-1500	-750	
← Carry over factor ← Carry over factor					
1000		-2500	2500	-4750	← RESULT

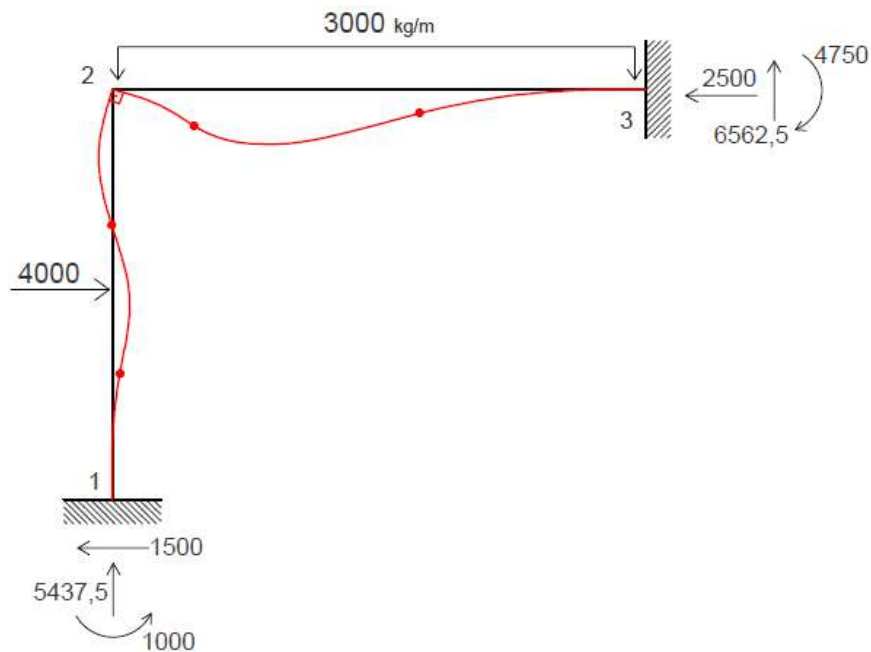
Once we get the results, we only need to apply the methodology observed in 5.1.1. section to obtain the results.



Diagrams



Structure's reactions and balance equation. Deformation



$$\Sigma H = 0; 4000 = 1500 + 2500; 4000 = 4000 \quad \checkmark$$

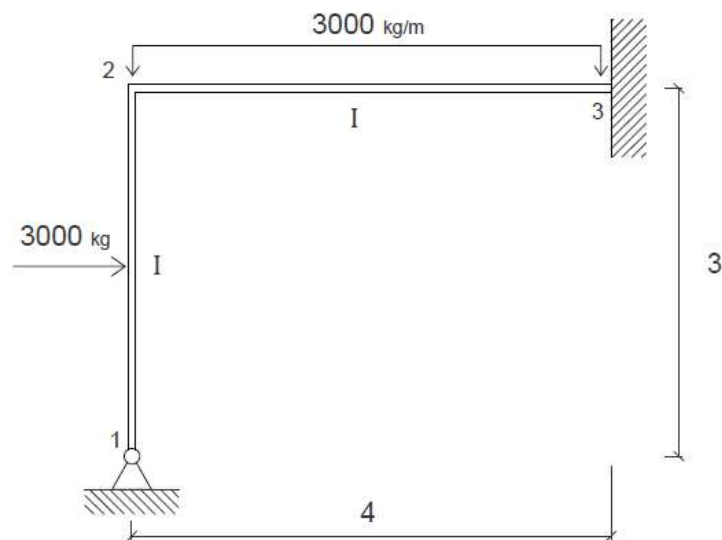
$$\Sigma V = 0; 3000 \cdot 4 = 5437,5 + 6562,5; 12000 = 12000 \quad \checkmark$$

$$\Sigma M_1 = 0; -4000 \cdot 1,5 - 3000 \cdot 4 \cdot 2 + 6562,5 \cdot 4 + 2500 \cdot 3 - 4750 + 1000 = 0$$

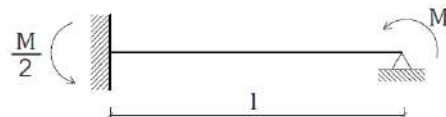
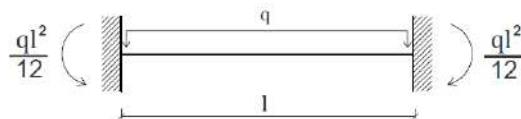
$$-6000 - 24000 + 26250 + 7500 - 4750 + 1000 = 0$$

$$34750 = 34750 \quad \checkmark$$

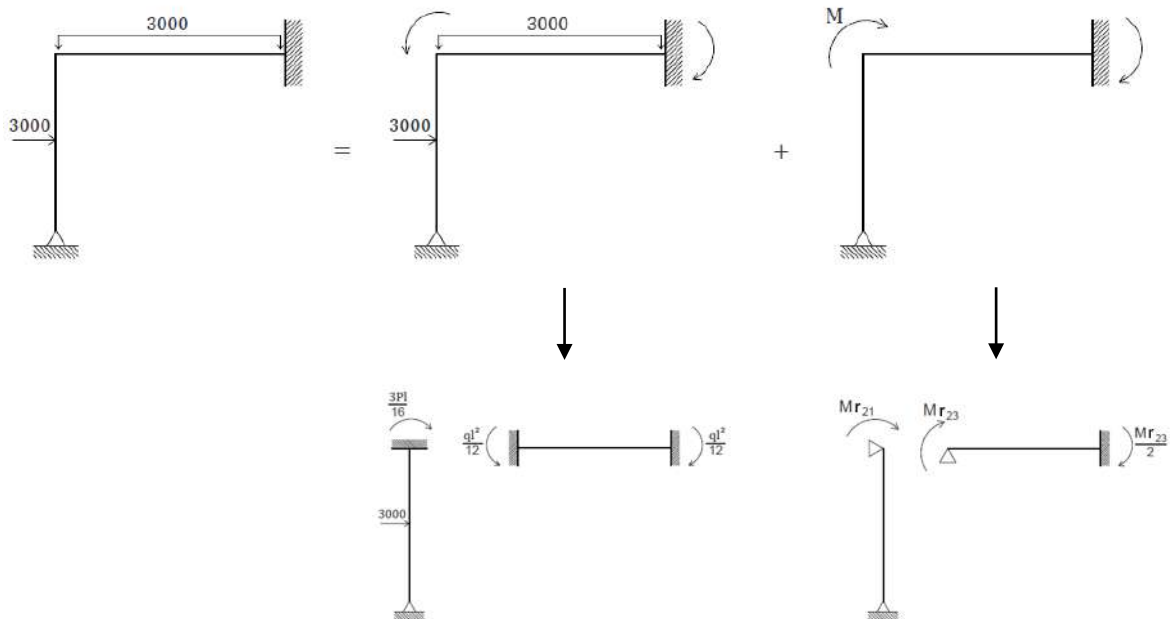
Let's calculate other similar frame to better understand the method. Solve the structure's reactions and draw moment, shear and axial strength diagrams. Draw the deformation.



Data:



Diagram



Fixed end moment

$$M_{23} = -M_{32} = \frac{ql^2}{12} = \frac{3000 \cdot 4^2}{12} = 4000 \text{ kgm}$$

$$-M_{21} = \frac{3Pl}{16} = \frac{3 \cdot 3000 \cdot 3}{16} = \frac{27000}{16} = 1687,5 \text{ kg} \cdot \text{m}$$

Stiffness

$$K_{21} = \frac{3EI}{l} = EI; K_{23} = \frac{4EI}{l} = EI$$

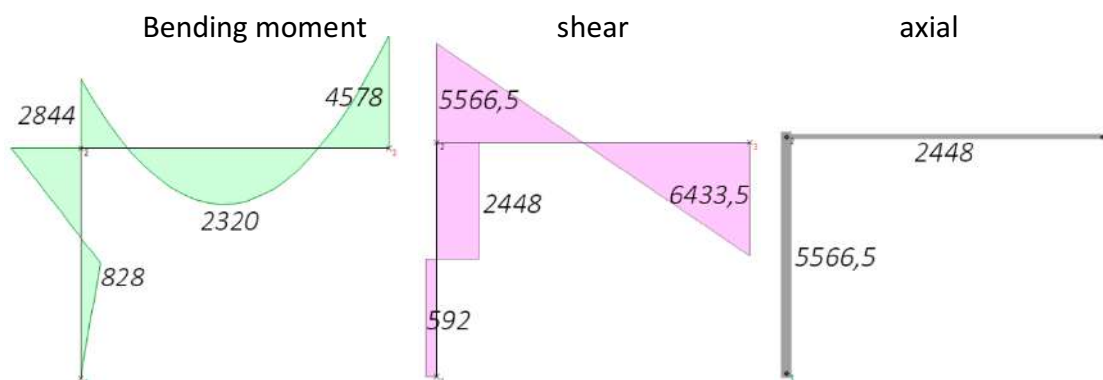
Distribution factors

$$r_{21} = \frac{K_{21}}{K_{21} + K_{23}} = \frac{EI}{2EI} = 0,5; r_{23} = 1 - 0,5 = 0,5$$

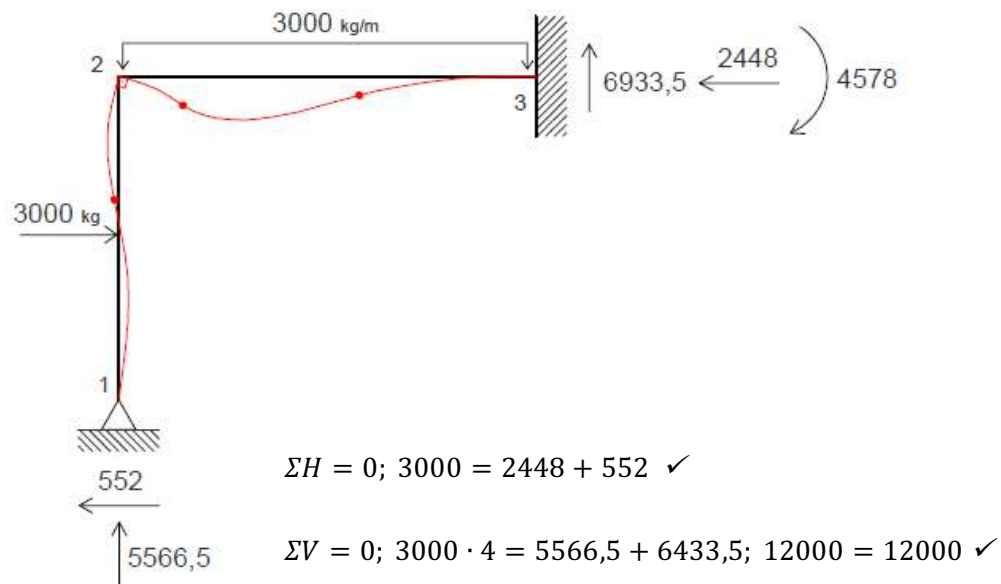
Cross table

1	2			3	← Nodes / Supports
12		21	23	32	← Bar ends
-		0,5	0,5	-	← Distribution factors
	2312,5	-1687,5	4000	-4000	← FEM
	-2312,5	-1156,25	-1156,8	-750	
0		-2844	2844	-4578	← Result

Diagrams



Reactions and deformation

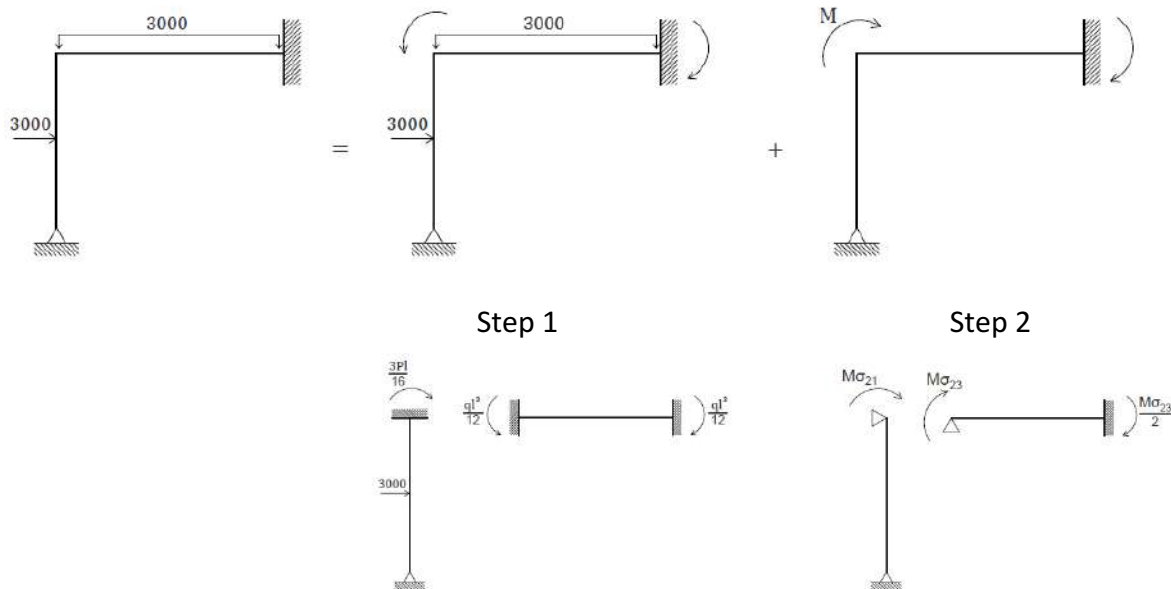


$$\Sigma M_1 = 0; -3000 \cdot 1,5 - 3000 \cdot 4 \cdot 2 + 2448 \cdot 3 + 2500 \cdot 3 + 6433,5 \cdot 4 - 4578 = 0$$

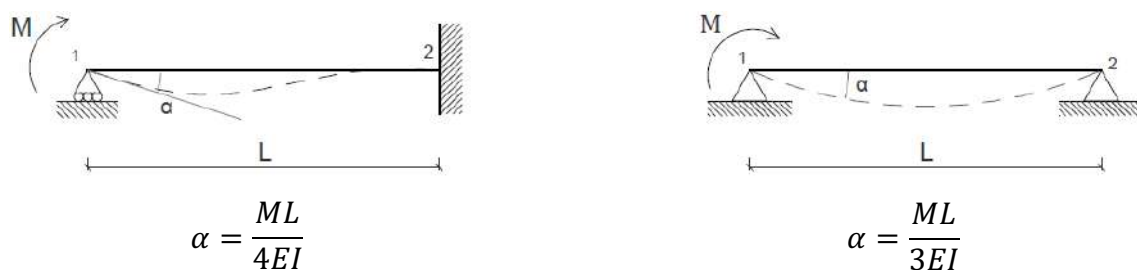
$$0 = 0 \quad \checkmark$$

5.1.7. Calculation of rotation in nodes

As we can see in the diagram of the previous frame, between the different steps of the application of the principle of superposition, the only one where the nodes experiment a rotation is when applying the carry over factors, in step 2:



Taking into account the way in which we apply the principle of superposition when using Cross method, we always have one of these two situations in step 2 (Section 5.1.3):

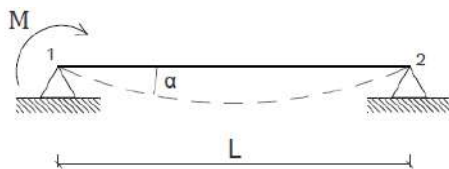


Therefore, the process to calculate rotations is: take the moment which makes the node rotate from the table (the moment applied when using the distribution factors), and introduce it in the correct formula from the two above.

Following with the same frame, let us calculate the rotation in the second node. If we analyse the Cross table, we can underline the moments that make rotate the node:

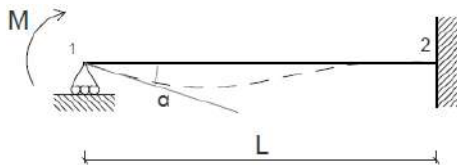
1	2			3	← Nodes / Supports
12		21	23	32	← Bar ends
-		0,5	0,5	-	← Distribution factors
	2312,5	-1687,5	4000	-4000	← FEM
	-2312,5	-1156,25	-1156,8	-750	
0		-2844	2844	-4578	← Result

Calculation of the rotation in 2-1 bar



$$\alpha = \frac{ML}{3EI} = \frac{-1156,25 \times 3}{3EI} = \frac{-1156,25}{EI} \text{ Rad}$$

Calculation of the rotation in 2-3 bar



$$\alpha = \frac{ML}{4EI} = \frac{-1156,25 \times 4}{4EI} = \frac{-1156,25}{EI} \text{ Rad}$$

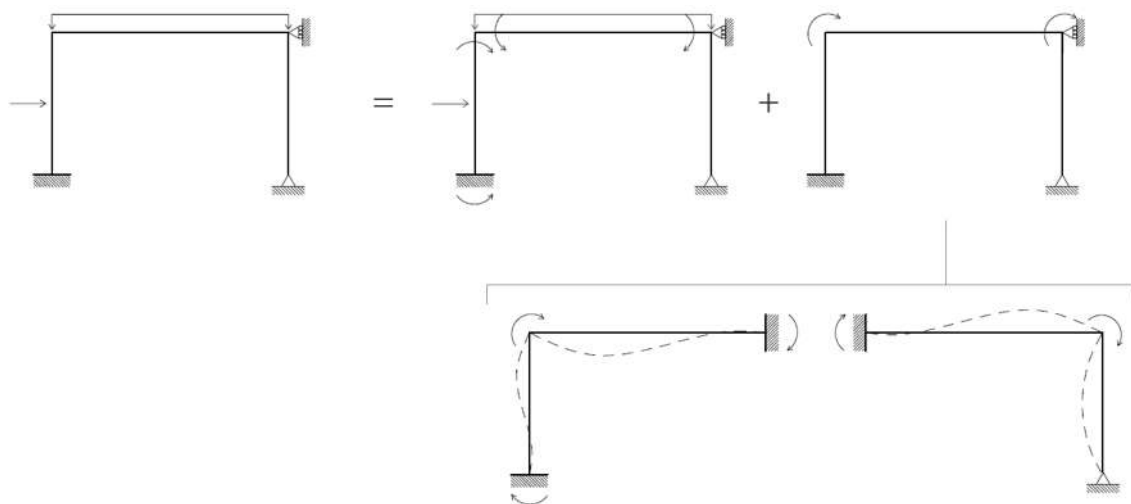
By calculating the rotation in the two bars, we verify that every bar's end leading to the same stiff node, experience the same rotation.

5.1.8. Non sway orthogonal frames. Multiple nodes.

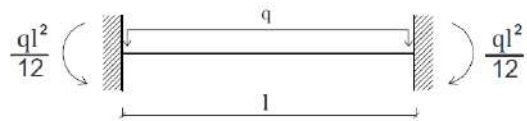
In the previous examples we have solved single node frames. It means that when we apply the moment that releases the node, the opposite ends of the bars leading to the node are supports (fixed or hinged support). In this case we can apply the stiffness seen in 5.1.3 section.

But when we have more than one node, it changes. The opposite end of a rotating node can be another node, not a fixed or hinged support. So that doesn't happen, we release nodes individually. It means that when letting one node rotate, we apply the necessary moment to restrain the rotation in the adjacent node (as we do in the first step of the method with the FEM). Like that, we can analyse the adjacent node as a fixed support, and we can apply the stiffness we know.

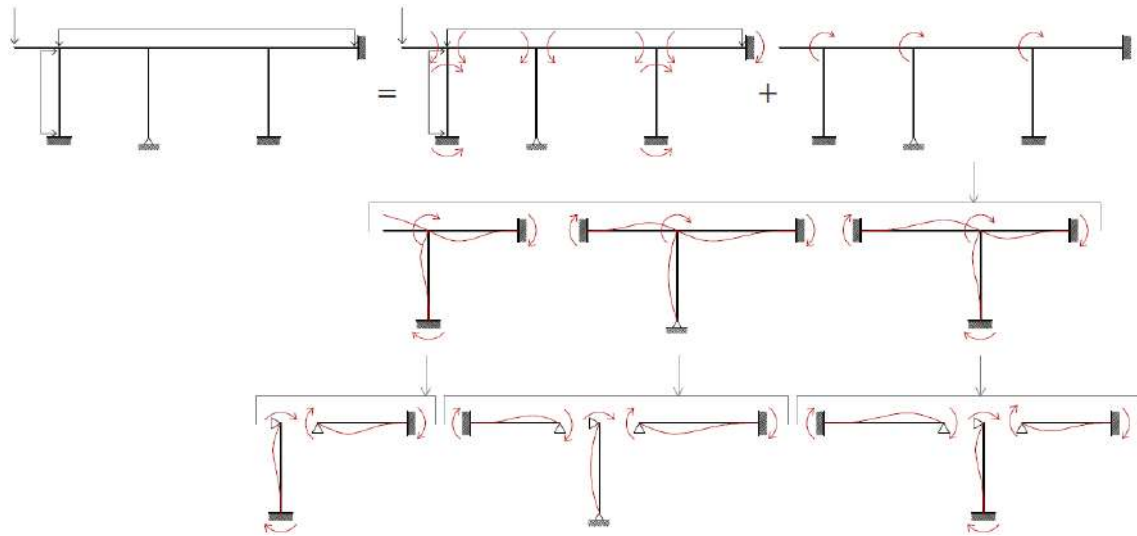
Later on, when we release the other nodes, apart from the FEM of the first step, we will take into account too the moment applied previously to restrain the rotation. The following diagram represents this process:



In the next exercise we will solve a three nodes frame, and we will better understand the previous theory through the practice.

$$EI = \text{constant}; E = 2 \cdot 10^5 \text{ kg/cm}^2; I = 1,25 \cdot 10^5 \text{ cm}^4$$


Diagram



Fixed end Moment (FEM)

$$M_{32} = M_{23} = \frac{ql^2}{12} = \frac{2000 \cdot 3^2}{12} = 1500 \text{ kg} \cdot \text{m}$$

$$M_{35} = M_{53} = \frac{ql^2}{12} = \frac{3000 \cdot 3^2}{12} = 2250 \text{ kg} \cdot \text{m}$$

$$M_{57} = M_{75} = \frac{ql^2}{12} = \frac{3000 \cdot 4^2}{12} = 4000 \text{ kg} \cdot \text{m}$$

$$M_{78} = M_{87} = \frac{ql^2}{12} = \frac{3000 \cdot 3^2}{12} = 2250 \text{ kg} \cdot \text{m}$$

$$M_1 = P \cdot l = 1000 \cdot 1,5 = 1500 \text{ kg} \cdot \text{m}$$

Stiffness

$$K_{32} = K_{76} = \frac{4EI}{l} = \frac{4EI}{3} = 1,3EI$$

$$K_{54} = \frac{3EI}{l} = \frac{3EI}{3} = EI$$

$$K_{35} = K_{78} = \frac{4EI}{l} = \frac{4EI}{3} = 1,3EI$$

$$K_{57} = \frac{4EI}{l} = \frac{4EI}{4} = EI$$

Distribution factors

3. node:

$$r_{32} = \frac{K_{32}}{K_{32} + K_{35}} = \frac{1, \hat{3}EI}{1, \hat{3}EI + 1, \hat{3}EI} = 0,5$$

$$r_{35} = \frac{K_{35}}{K_{32} + K_{35}} = 0,5$$

5. node:

$$r_{53} = \frac{K_{53}}{K_{53} + K_{54} + K_{57}} = \frac{1, \hat{3}EI}{1, \hat{3}EI + EI + EI} = \frac{1, \hat{3}}{3, \hat{3}} = 0,4$$

$$r_{54} = \frac{K_{54}}{K_{53} + K_{54} + K_{57}} = \frac{EI}{3, \hat{3}EI} = 0,3$$

$$r_{57} = \frac{K_{57}}{K_{53} + K_{54} + K_{57}} = \frac{EI}{3, \hat{3}EI} = 0,3$$

7. node:

$$r_{75} = \frac{K_{75}}{K_{75} + K_{76} + K_{78}} = \frac{EI}{EI + 1, \hat{3}EI + 1, \hat{3}EI} = \frac{EI}{3, \hat{6}} = 0,27$$

$$r_{76} = \frac{K_{76}}{K_{75} + K_{76} + K_{78}} = \frac{1, \hat{3}EI}{3, \hat{6}EI} = 0,365$$

$$r_{78} = \frac{K_{78}}{K_{75} + K_{76} + K_{78}} = \frac{1, \hat{3}EI}{3, \hat{6}EI} = 0,365$$

Cross Table

2	3			5			7			4	6	8
23		32	31	35				75	76	78	67	87
-		0,500	-	0,500				0,273	0,364	0,364	-	-
1500,00	-750,00	-1500,00	-1500,00	2250,00				-4000,00		2250,00		-2250,00
187,50	750,00	375,00		375,00		1937,50						
	-387,50			-387,50		-1937,50						
96,88	387,50	193,75		193,75		375,14		2040,63	742,05	742,05	371,02	371,02
	-75,03			-75,03		-375,14		-56,27				
18,76	75,03	37,51		37,51		26,43		56,27	20,46	20,46	10,23	10,23
						-26,43		-3,96				
								3,96	1,44	1,44	0,72	0,72
1803,13		-893,74	-1500,00	2393,74				-3777,90	763,95	3013,95	381,97	-1868,03

Moment values:

$$M_{23} = 1800 \text{ kg} \cdot \text{m}$$

$$M_{32} = -900 \text{ kg} \cdot \text{m}$$

$$M_{35} = 2400 \text{ kg} \cdot \text{m}$$

$$M_{53} = -2880 \text{ kg} \cdot \text{m}$$

$$M_{54} = -700 \text{ kg} \cdot \text{m}$$

$$M_{45} = 0 \text{ kg} \cdot \text{m}$$

$$M_{57} = 3580 \text{ kg} \cdot \text{m}$$

$$M_{75} = -3785 \text{ kg} \cdot \text{m}$$

$$M_{76} = 767 \text{ kg} \cdot \text{m}$$

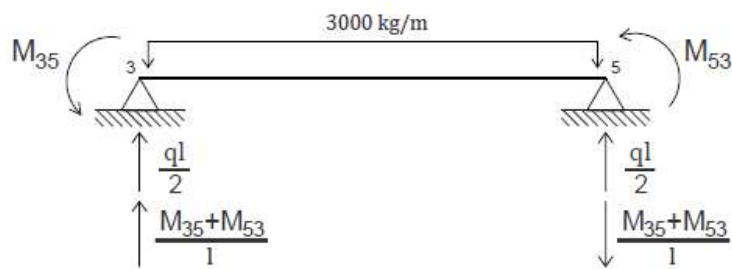
$$M_{67} = 383 \text{ kg} \cdot \text{m}$$

$$M_{78} = 3018 \text{ kg} \cdot \text{m}$$

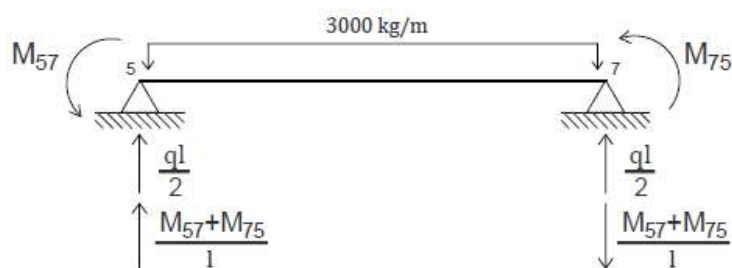
$$M_{87} = -1867 \text{ kg} \cdot \text{m}$$

$$M_{31} = -1500 \text{ kg} \cdot \text{m}$$

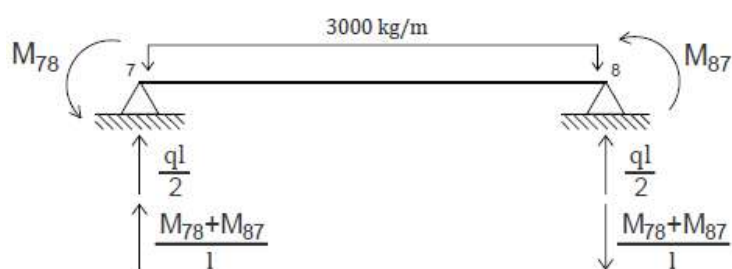
Beams



3-5 bar



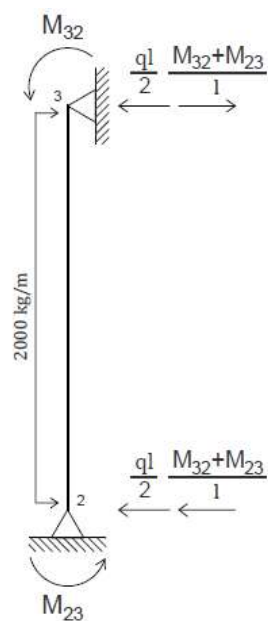
5-7 bar



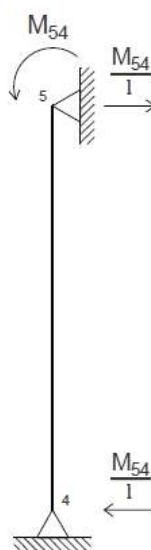
7-8 bar

Columns

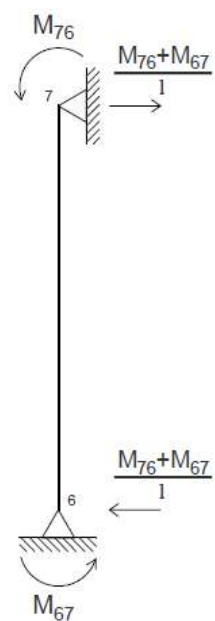
3-2 bar



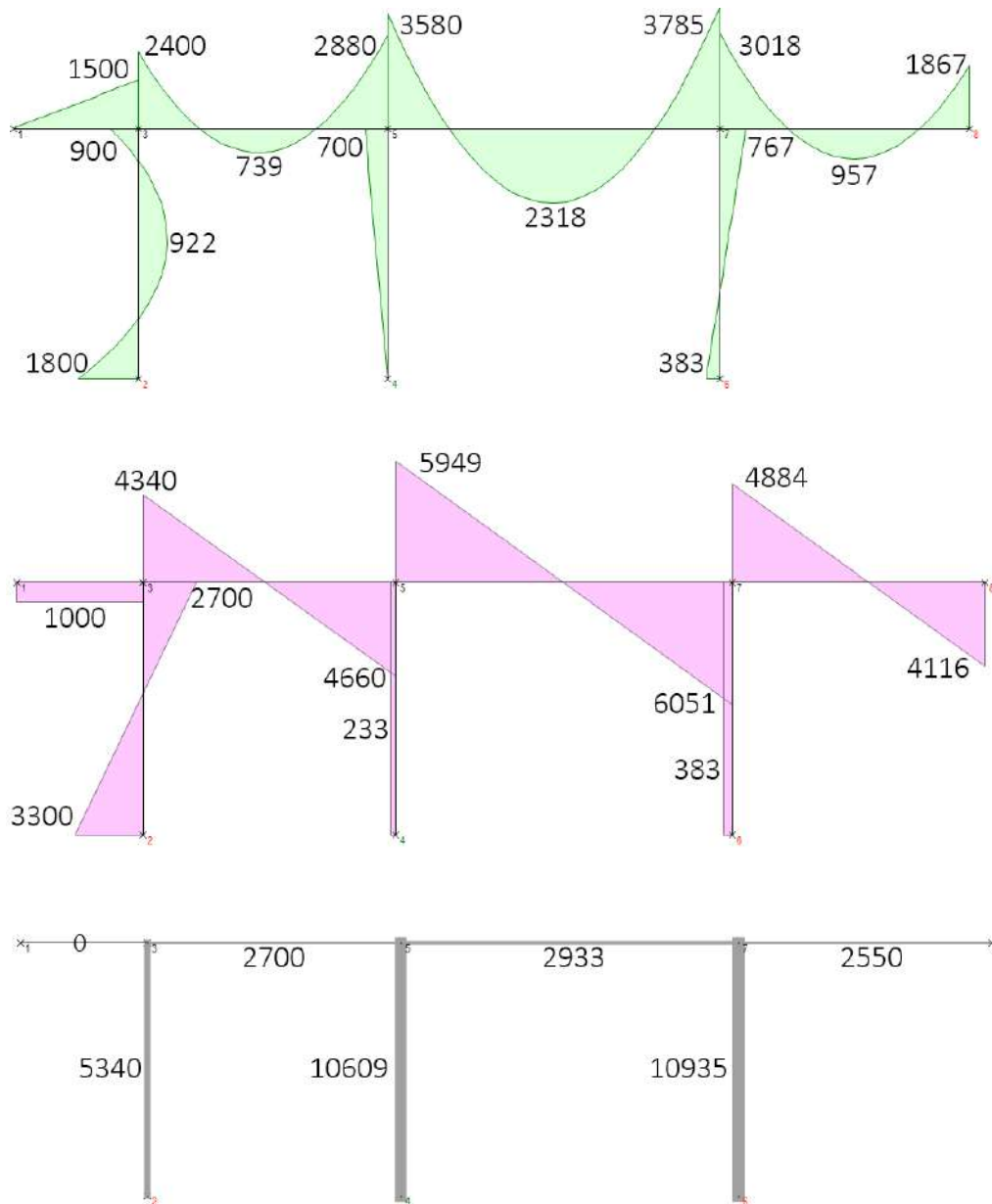
5-4 bar



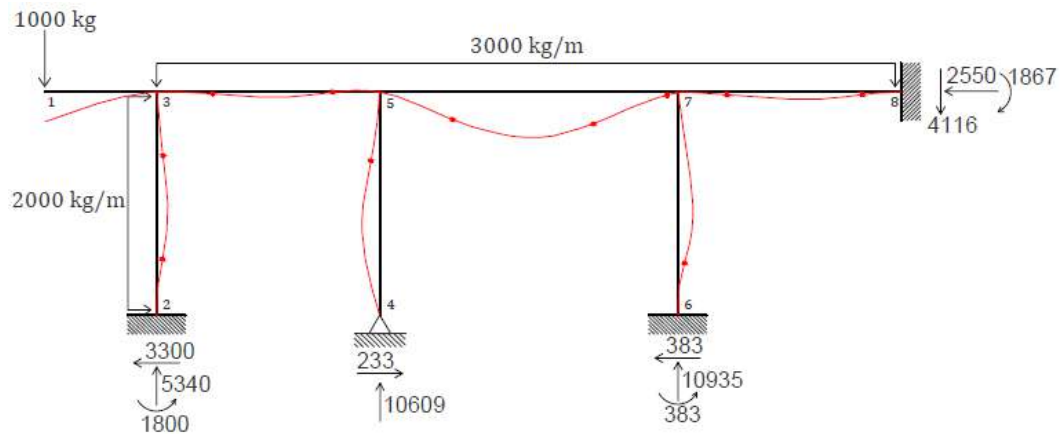
7-6 bar



Diagrams (by rounding table results)



Reactions and deformation



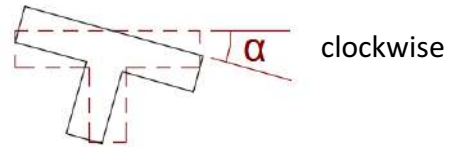
$$\Sigma H = 0; 2000 \cdot 3 = 3300 - 233 + 383 + 2550; 6000 = 6000 \checkmark$$

$$\Sigma V = 0; 1000 + 3000 \cdot 10 = 5340 + 10609 + 10935 + 4116; 31000 = 31000 \checkmark$$

$$\Sigma M_2 = 0; -3000 \cdot 10 \cdot 5 - 2000 \cdot 3 \cdot 1,5 + 1000 \cdot 1,5 + 10609 \cdot 3 + 10935 \cdot 7 + 4116 \cdot 10 + 2550 \cdot 3 + 1800 + 383 - 1867 = 0; -2 \cong 0 \checkmark$$

5. node rotation (calculated in 5-7 end)

$$\alpha = \frac{M \cdot l}{4EI} = \frac{-70275 \cdot 400}{4 \cdot 2,5 \cdot 10^{10}} = -2,811 \cdot 10^{-4} \text{ rad}$$



$$M = -581,5 - 113,5 - 7,75 = -702,75 \text{ kg} \cdot \text{m} = -70275 \text{ kg} \cdot \text{cm}$$

$$EI = 2 \cdot 10^5 \cdot 1,25 \cdot 10^5 = 2,5 \cdot 10^{10} \text{ kg} \cdot \text{cm}^2$$

5. joint node (calculated in 5-3 end)

$$\alpha = \frac{M \cdot l}{4EI} = \frac{-93020 \cdot 300}{4 \cdot 2,5 \cdot 10^{10}} = 2,80 \cdot 10^{-4} \text{ rad}$$

$$M = -775 - 145 - 10,2 = -930,2 \text{ kg} \cdot \text{m} = -93020 \text{ kg} \cdot \text{cm}$$

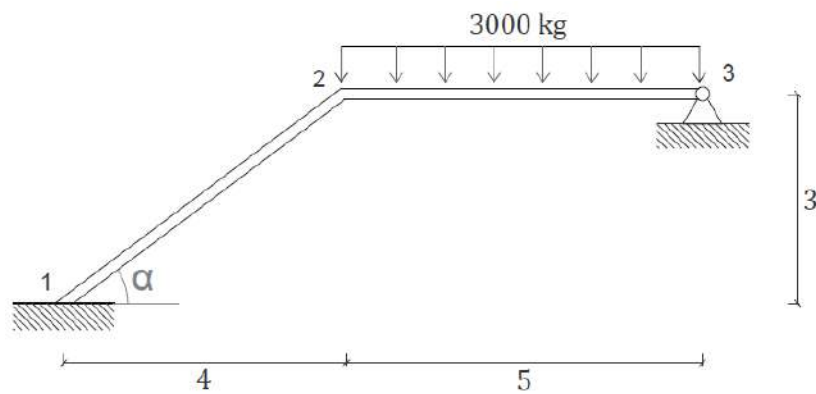
5. joint node (calculated in 5-4 end)

$$\alpha = \frac{M \cdot l}{3EI} = \frac{-70275 \cdot 300}{3 \cdot 2,5 \cdot 10^{10}} = 2,80 \cdot 10^{-4} \text{ rad}$$

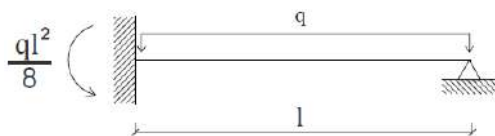
$$M = -581,5 - 113,5 - 7,75 = -702,75 \text{ kg} \cdot \text{m} = -70275 \text{ kg} \cdot \text{cm}$$

5.1.9. Inclined bar frames. Exercise

EXERCISE. Using Cross method, solve the structure's reactions and draw moment, shear and axial strength diagrams. Draw the deformation.



Data:



Fixed end Moment (FEM)

$$M_{23} = \frac{ql^2}{8} = \frac{3000 \cdot 5^2}{8} = 9375 \text{ kg} \cdot \text{m}$$

Stiffness

$$K_{21} = \frac{4EI}{l} = 0,8 EI ; K_{23} = \frac{3EI}{l} = 0,6 EI$$

Distribution factors

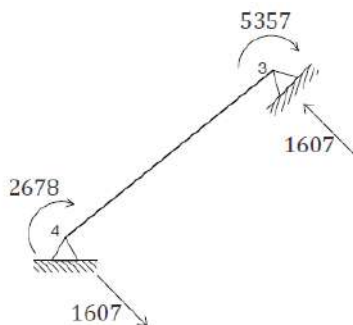
$$r_{21} = \frac{K_{21}}{K_{21} + K_{23}} = \frac{0,8 EI}{(0,8 + 0,6)EI} = 0,57$$

$$r_{23} = \frac{K_{23}}{K_{21} + K_{23}} = \frac{0,6 EI}{(0,8 + 0,6)EI} = 0,43$$

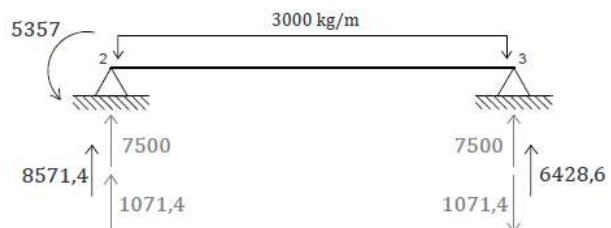
Cross Table

1	2			3
12		21	23	32
-		0,57	0,43	-
	9375		9375	
-2678	-9375	-5357	-4018	
-2678		-5357	5357	

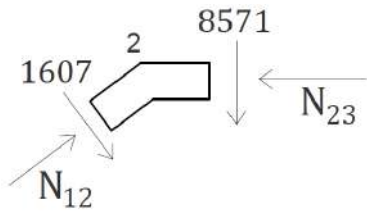
1-2 bar



2-3 bar



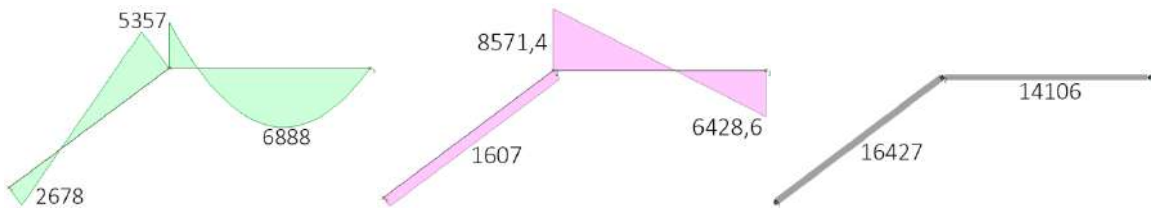
Node 2 equilibrium



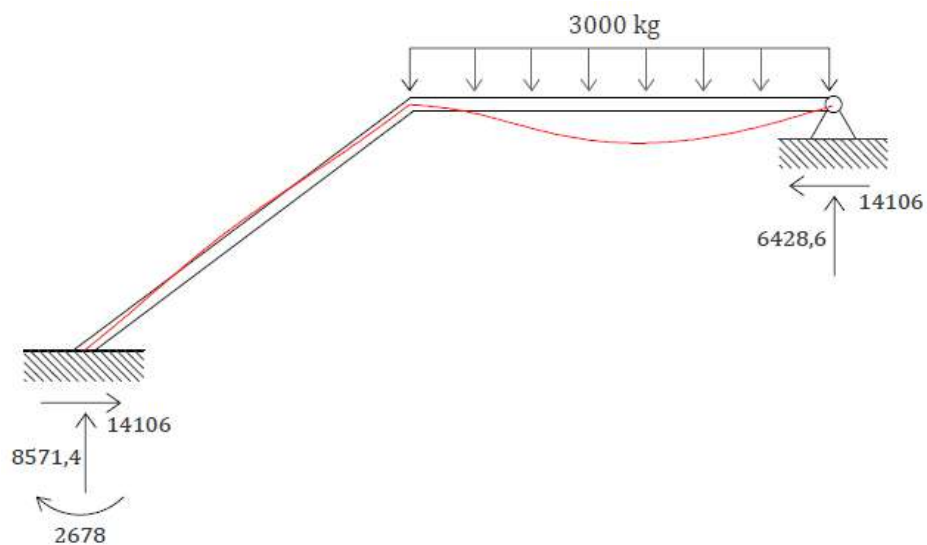
$$\Sigma V = 0 ; N_{21} \cdot 0,6 = 1607 \cdot 0,8 + 8571 ; N_{21} = 16427$$

$$\Sigma H = 0 ; N_{21} \cdot 0,8 + 1607 \cdot 0,6 = N_{23} ; N_{23} = 14106$$

Diagrams



Reactions and deformation



$$\Sigma H = 0 ; 14106 = 14106 \quad \checkmark$$

$$\Sigma V = 0 ; 3000 \cdot 5 = 6428,6 + 8571,4 ; 15000 = 15000 \quad \checkmark$$

$$\Sigma M_1 = 0 ; 6428,6 \cdot 9 + 14106 \cdot 3 = 15000 \cdot 6,5 + 2678$$

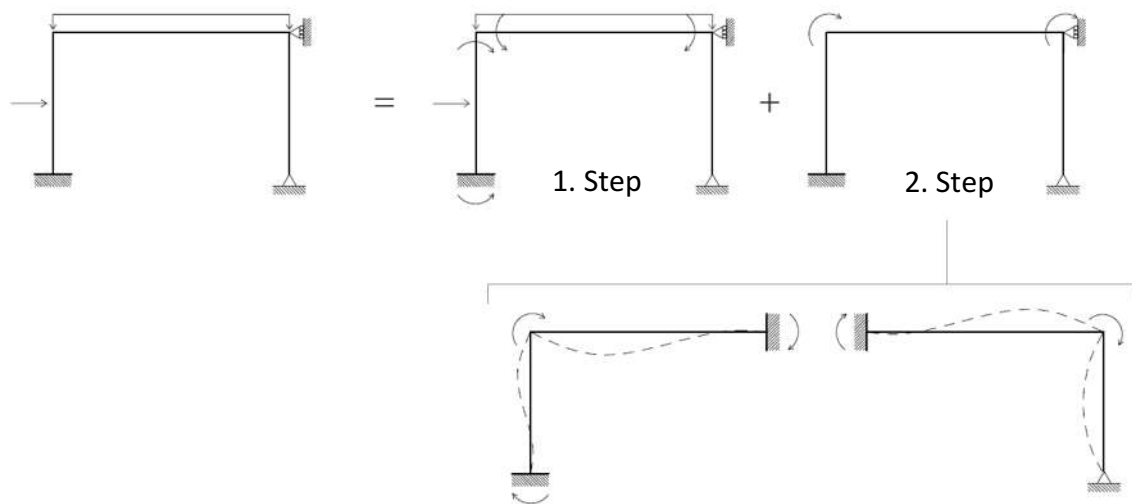
$$57857,4 + 42318 = 97500 + 2678 ; 100178 = 100178 \quad \checkmark$$

5.2. SWAY FRAMES

5.2.1. Introduction, methodology and previous concepts

In the previous sections of these notes we have studied frames where nodes can't move (it should be remembered that this assertion is correct on the condition that bars are inextensible). In the present section we will describe how to proceed when frame's nodes can experience displacements (sway frames).

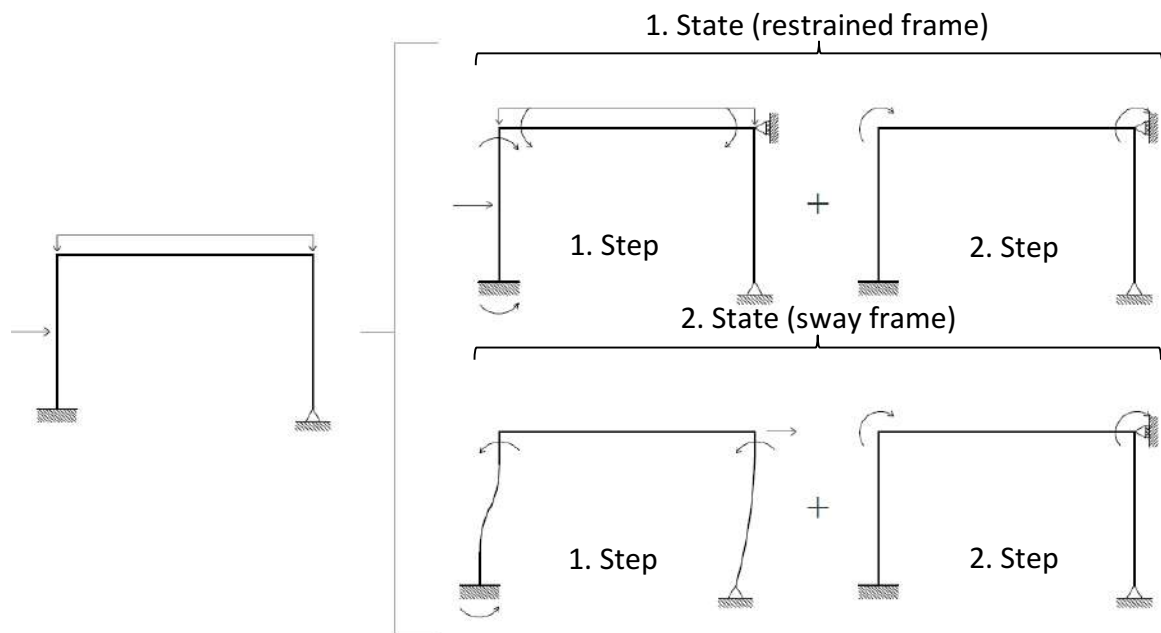
In 5.1.8 section we have analysed this frame:



This diagram is right because the hinged support located in the upper-right node restrains the displacement. If we remove this support the frame would become sway, and the diagram wouldn't fulfil the superposition principle anymore, because steps 1 and 2 wouldn't reflect the existing displacement in the real frame. In the same way, the reaction existing in this support in step 1 and 2 couldn't exist in this hypothetical frame.

To overcome this problem, we need to introduce an additional state in the superposition diagram with the ability of representing the displacement and removing the reactions of the inexistent support.

Following with the same frame, let us analyse the next diagram:



As we can see, we have added a second state similar to the first. Both of them have two steps, and the second step is equal in the two states. The main characteristics of each state are:

State 1

It represents a restrained frame. The case we have studied in the previous sections of these notes. In the first step we restrain rotations in nodes by using the Fixed End Moments, and in the second one we release the nodes by applying moments with the same value, but in opposite direction.

State 2

It represents a sway frame. In the first step of this new stage we solve the problem of the displacement, by introducing a displacement with null rotation in nodes. It means that when the displacement is done, we apply Fixed End Moments which restrain the rotation. In the second step we release the nodes following the same methodology of the first state.

This diagram needs to respect two conditions to fulfil the superposition principle:

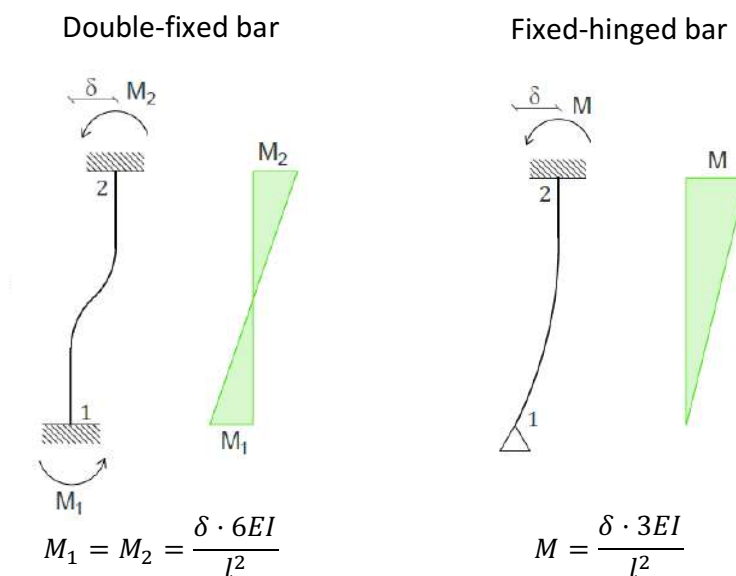
- We need to calculate the fixed end moments to restrain the rotation during the translation
- All the actions and reactions applied in the upper-right node must add up to zero.

For fulfilling those two conditions we have to introduce three new concepts that will be developed in the following sections:

- Fixed End Moments of displacements
- Translational degree of freedom
- Displacements equilibrium equations

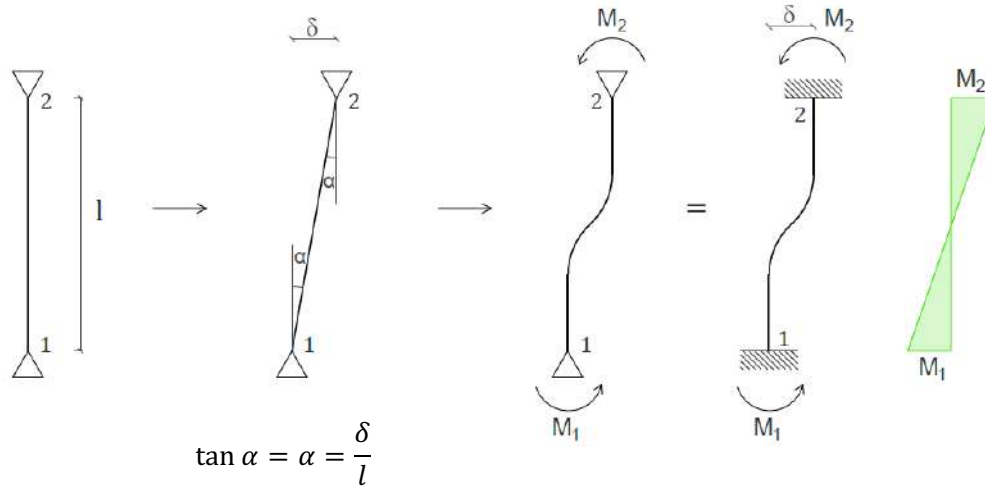
5.2.2. Fixed End Moments of displacements

In the context of the Cross method, the fixed end moments of displacements are the bending moments created in fixed ends of simple bars when there is a displacement between the ends of the bar. Therefore, it is the relationship between the displacement, and the fixed end moment which is caused. We have these two cases:

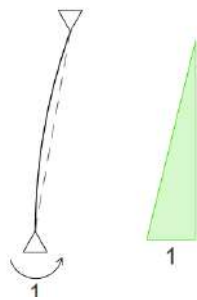


We can obtain the results above by using different methods. In the following pages, we will check those values by using the Unit Load method.

-Double fixed end bar. Real system:

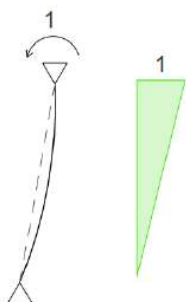


Unit system 1:



R	 + 
U	$\alpha_1 = \frac{1}{EI} \left(\frac{1}{3} M_1 \cdot l - \frac{1}{6} \cdot M_2 \cdot l \right)$

Unit system 2:



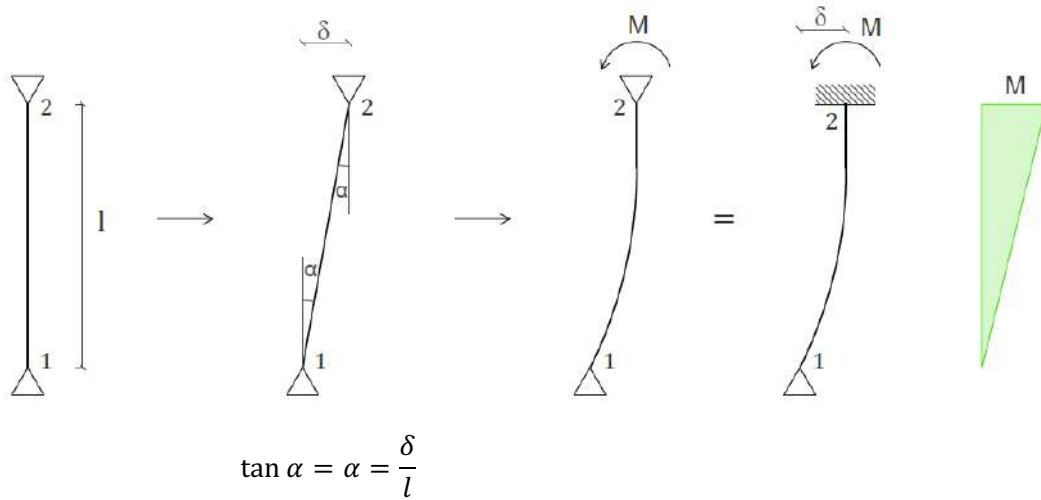
R	 + 
U	$\alpha_2 = \frac{1}{EI} \left(-\frac{1}{6} M_1 \cdot l + \frac{1}{3} \cdot M_2 \cdot l \right)$

$$\alpha_1 = \alpha_2 = \alpha = \frac{M_1 l}{3} - \frac{M_2 l}{6} = \frac{-M_1 l}{6} + \frac{M_2 l}{3} \rightarrow \frac{M_1}{2} = \frac{M_2}{2} \rightarrow \boxed{M_1 = M_2}$$

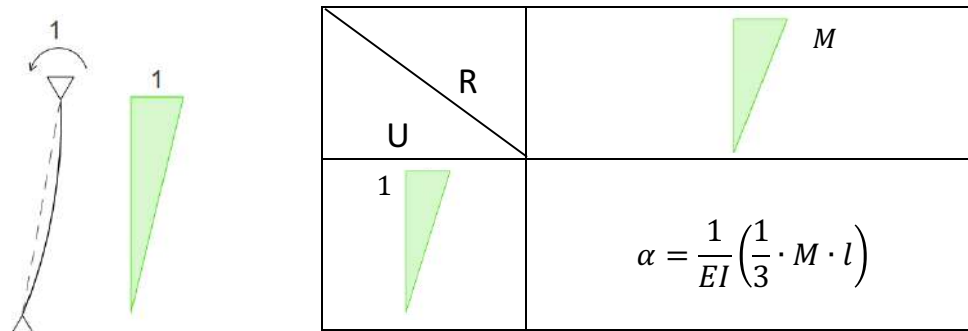
$$\left. \begin{aligned} \alpha &= \frac{1}{EI} \left(\frac{1}{3} \cdot M \cdot l - \frac{1}{6} \cdot M \cdot l \right) = \frac{M \cdot l}{6EI} \\ \alpha &= \frac{\delta}{l} \end{aligned} \right\} \alpha = \frac{\delta}{l} = \frac{M \cdot l}{6EI} \rightarrow \boxed{\delta = \frac{M l^2}{6EI}; M = \frac{\delta \cdot 6EI}{l^2}}$$

-Fixed -hinged bar

Real system



Unit system:

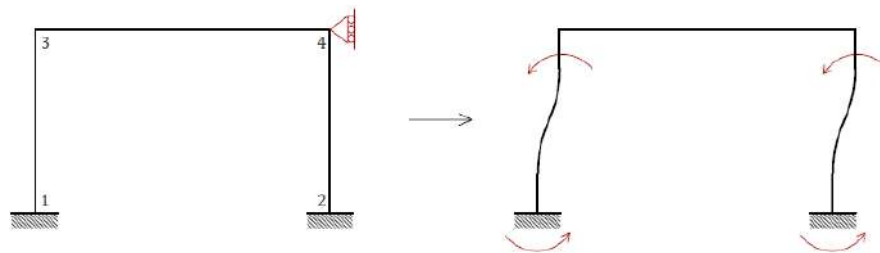


$$\left. \begin{array}{l} \alpha = \frac{M \cdot l}{3EI} \\ \alpha = \frac{\delta}{l} \end{array} \right\} \quad \frac{\delta}{l} = \frac{M \cdot l}{3EI} \rightarrow \boxed{\delta = \frac{Ml^2}{3EI}; M = \frac{\delta \cdot 3EI}{l^2}}$$

5.2.3. Translational degree of freedom

The translational degree of freedom is the number of independent displacements that the nodes of a frame may experience. In other words, it is the number of reactions we will need to add to the frame to restrain it.

In the case of orthogonal frames, the translational degree of freedom can be guessed intuitively.



In this example we can observe that by putting a horizontal support on the node 4, the frame gets restrained. If we do not have this support, nodes 3 and 4 can move.

But in cases where it isn't easy to guess the number of reactions needed to restrain the frame, we can use the following formula:

$$d = 2n - b - r$$

d: translational degree of freedom

n: number of nodes

r: number of reactions that restrain displacements (vertical or horizontal reactions. Fixed end moments don't restrain displacements, but rotations)

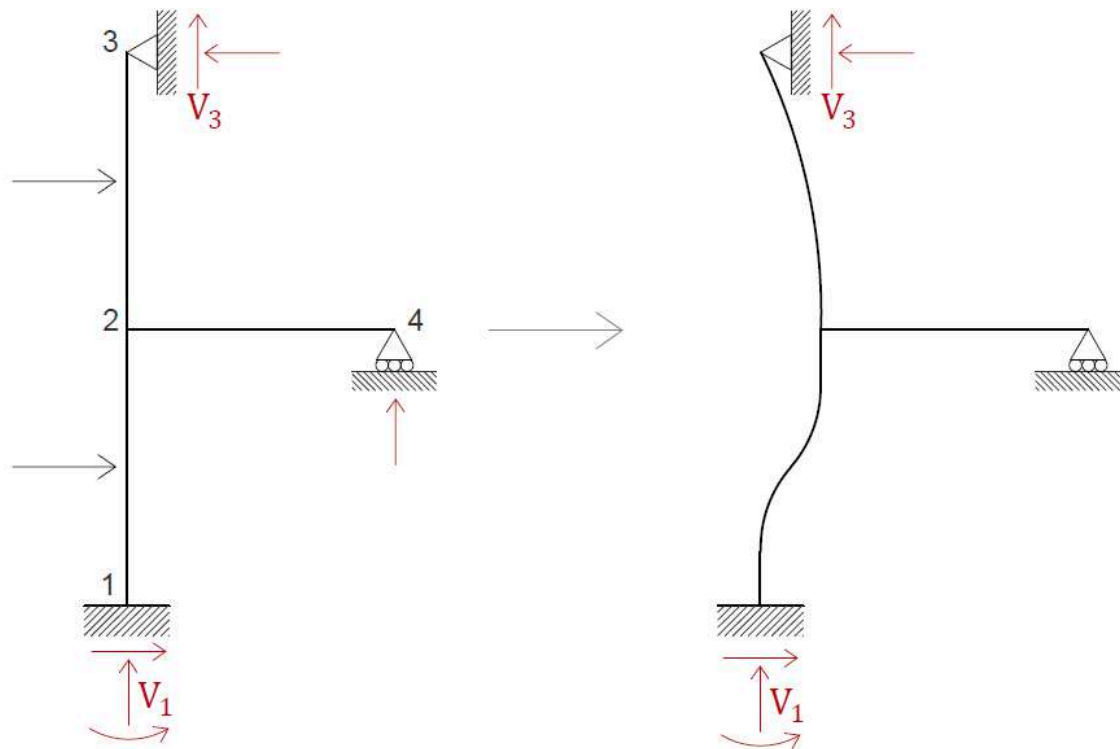
b: number of bars

Applying this formula in the same example:

$$d = 2 \cdot 4 - 3 - 4 = 8 - 7 = 1 \text{ (translational degree} \rightarrow 1)$$

When calculating the number of reactions that restrain displacements "r" we have to be careful. Depending on the geometry of the frame and the location of reactions, it might happen that some of them are redundant. It means that two or more reactions restrain the same displacement. In this case we have to remove the redundant reaction from the formula.

Let's analyse the following frame to better understand what redundant reactions are:

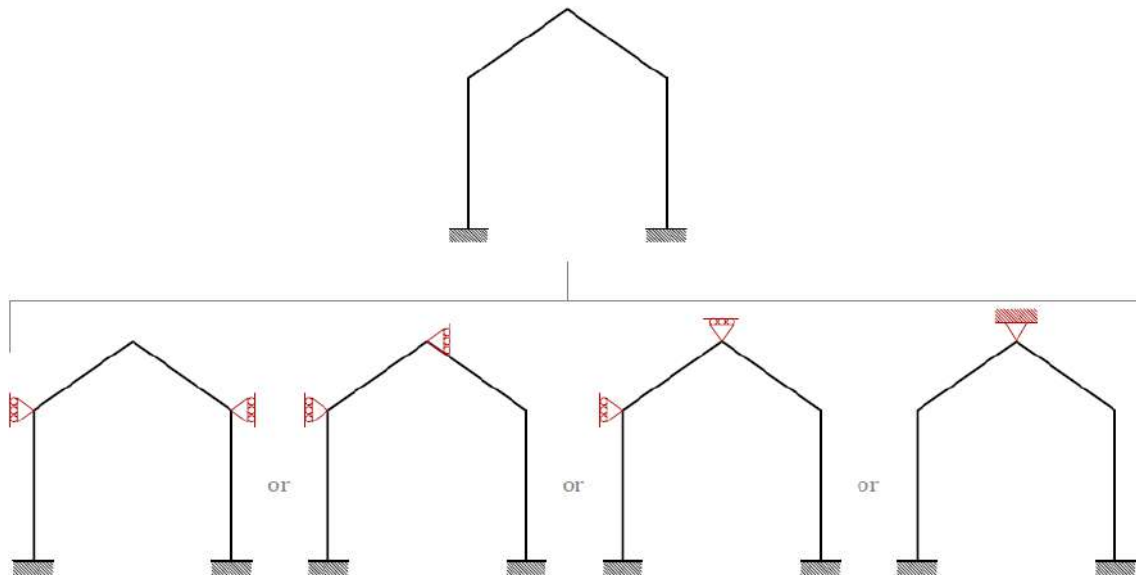


In this example we can observe that by using the formula, the translational degree is supposed to be zero, but intuitively we can guess that one displacement exists. That is because B1 and B3 reactions are redundant. Both of them restrain the vertical displacement of nodes 1-2-3, so when calculating the number of reactions “ r ”, we only need to take into account one of them. So in this frame $r = 4$.

In any case, the use of this formula may be helpful when we have doubts about the translational degree of a frame. But even knowing the translational degree we have to understand how the geometry of the frame get deformed, and we have to guess where to put the supports that restrain the frame.

It is important to note that even the translational degree of a frame is a fixed number, that is to say, that the amount of reactions we need to restrain it doesn't change, we can have different combinations to locate them.

The next frame's translational degree of freedom is 2. But, as we can see in the diagram, we have many options to locate the 2 reactions we need to restrain it.



It is very important remembering that when analysing the translational degree of freedom, we consider bars inextensible.

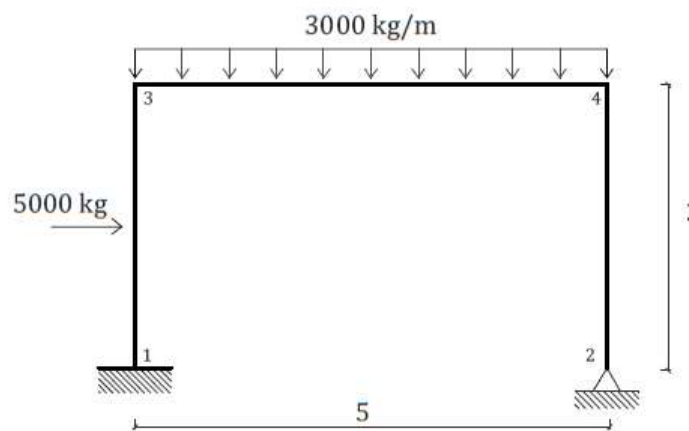
5.2.4. Displacements equilibrium equations

Displacement equilibrium equations are partial equilibrium equations we built by splitting the part of a frame containing the nodes that have been displaced in a translation.

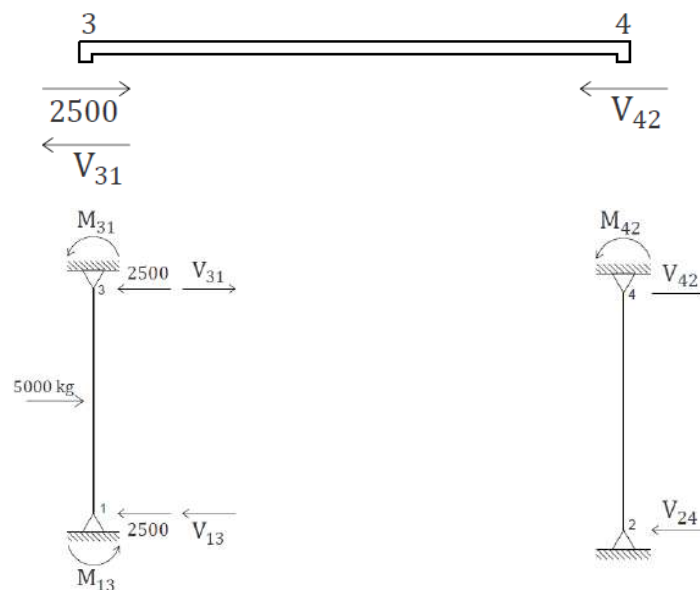
In any partial equilibrium equation we can have three kinds of elements: actions, reactions and internal strengths. The goal in cross method's equilibrium equations is to ensure that the only unknowns that are involved in the equations are bending moments.

As we will see in the following examples, we have to avoid involving axial strengths and reactions (except bending moment reactions) in the equations. Shear strength is not a problem, because we will observe that it is easy to connect the shear strength of a section with the bending moment of the bar it belongs.

Let's continue with the same example of the section 5.2.1:



The translational degree of freedom is 1, and the nodes involved in the displacement are 3 and 4. Therefore, we have to split the structure by cutting 3-1 and 4-2 ends, and build an equilibrium equation:



In this case, we have built the horizontal forces equilibrium equation ($\Sigma H = 0$), because that way, the only forces involved in the equation are shear strengths. And, as we will see, they are easy to connect with bending moments. Consequentially we have not drawn the distributed load of 3-4 bar, and the axial and bending moment strengths in sections 3-1 and 4-2, because they are not relevant for the horizontal forces equilibrium equation.

Therefore:

$$\Sigma H = 0; 2500 = V_{31} + V_{42}$$

By equilibrium in bars 3-1 and 4-2 we can connect V_{31} and V_{42} with bending moments:

$$V_{31} = V_{13} = \frac{M_{31} + M_{13}}{l_{31}}$$

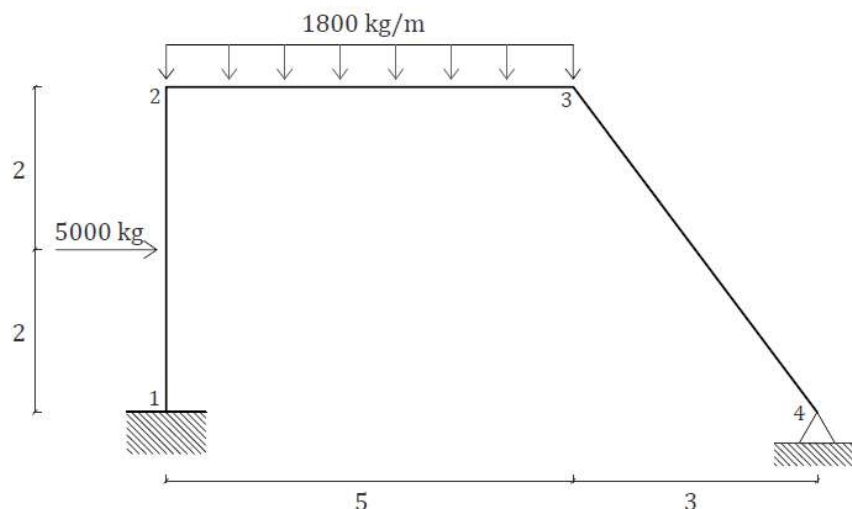
$$V_{42} = V_{24} = \frac{M_{42}}{l_{42}}$$

It should be noted that when approaching the equilibrium in bars 3-1 and 4-2 to deduce the value and direction of shear strengths, we draw the bending moments of the bar's ends counter clock-wise. The reason to do that is that they are still unknown, so until knowing their real value, we suppose that all of them act in this way.

Therefore:

$$\Sigma H = 0; 2500 = V_{31} + V_{42}; 2500 = \frac{M_{31} + M_{13}}{l_{31}} + \frac{M_{42}}{l_{42}}; \boxed{7500 = M_{31} + M_{13} + M_{42}}$$

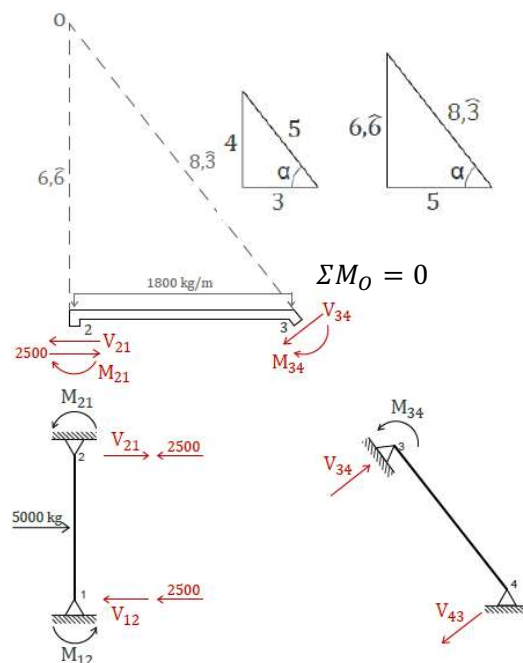
When we analyse frames with inclined bars, even conceptually the methodology is the same, displacements geometry and the approach to the displacement equilibrium equation becomes more complicated, as we might observe in the following example.



To decide what part of the frame we have to split to approach the equilibrium equation, we need to first analyse what the translational degree is, and what are the nodes involved in each displacement.

$$d = 2n - b - r = 2 \cdot 4 - 3 - 4 = 1 (\text{translational degree} = 1)$$

The displacement involves 2 and 3 nodes, so the equilibrium equation will be the next:



As we can see, we have split the structure in 2-1 and 3-4 sections, and we have approached the moments equilibrium equation in point 0 ($\Sigma M_O = 0$). The reason why we have decided to build this equation in this location is because it is the only way to avoid involving 2-1 and 3-4 section's axial strengths in the equation.

Therefore:

$$\Sigma M_O = 0$$

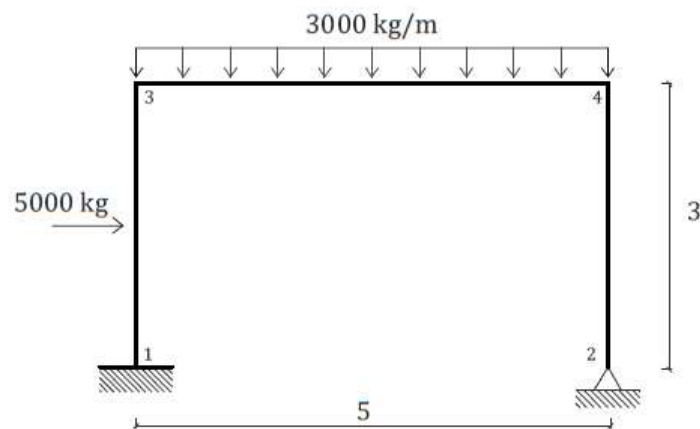
$$1800 \cdot 5 \cdot 2,5 - 2500 \cdot 6,6 + V_{21} \cdot 6,6 + V_{34} \cdot 8,3 + M_{21} + M_{34} = 0$$

$$5833,3 + 6,6 \frac{M_{21} + M_{12}}{4} + 8,3 \frac{M_{34}}{5} + M_{21} + M_{34} = 0$$

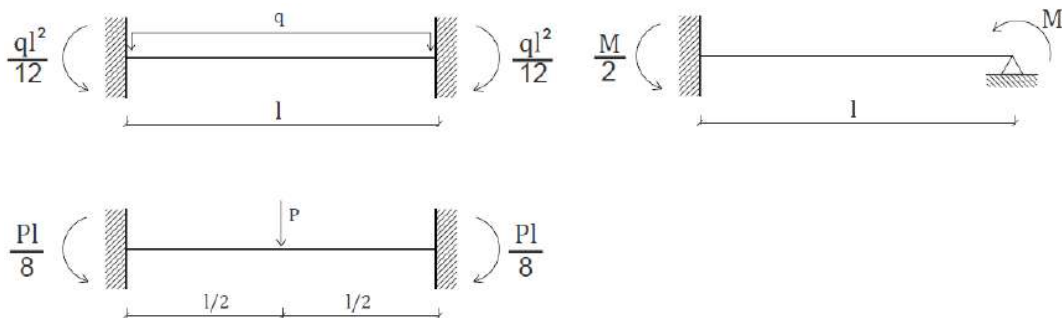
$$5833,3 + 2,6M_{21} + 1,6M_{12} + 2,6M_{34} = 0$$

5.2.5. Sway orthogonal frame's. Example with one displacement

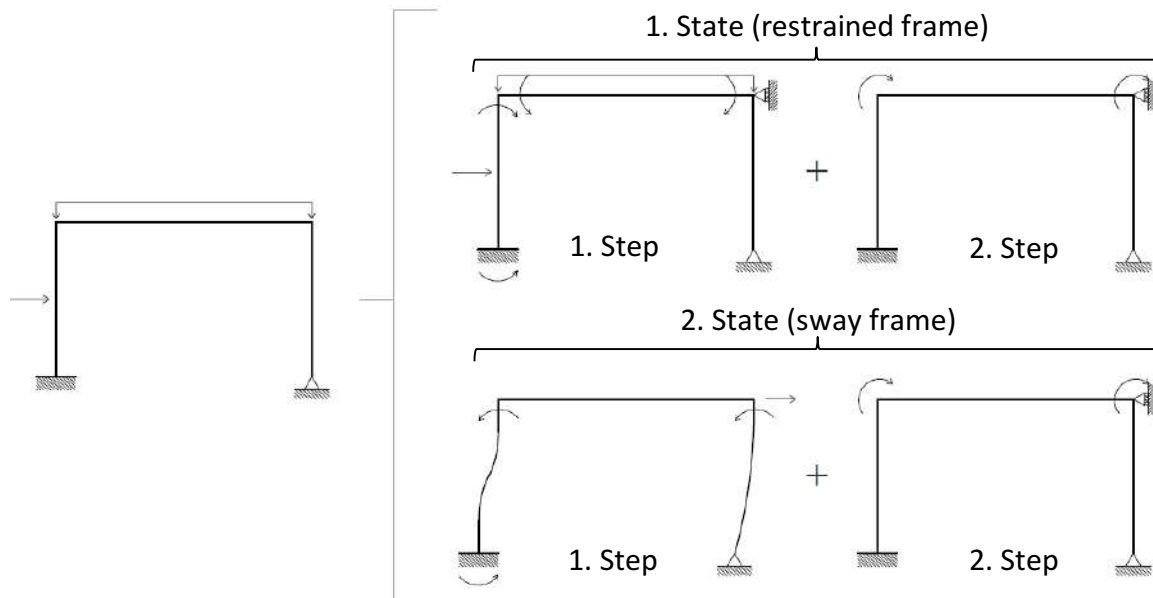
In this section we will solve a sway frame by Using the Cross method. We will calculate the reactions, and draw bending moment, shear and axial diagrams. Through this process of adapting cross method to translational frames we will understand the theory seen in previous sections.



Data: $EI = \text{constant} = 1,35 \cdot 10^{10} \text{ kg/cm}^2$



As we have seen in section 5.2.1, the diagram to follow for solving this frame is the next one:



To solve each state we have to approach a Cross table. Both of them will have the same stiffness and distribution factors, because the second step in each state is equal.

State 1. Fixed end Moment (FEM)

$$M_{13} = M_{31} = \frac{Pl}{8} = \frac{5000 \cdot 3}{8} = 1875 \text{ kg} \cdot \text{m}$$

$$M_{34} = M_{43} = \frac{ql^2}{12} = \frac{3000 \cdot 5^2}{12} = \frac{75000}{12} = 6250 \text{ kg} \cdot \text{m}$$

Stiffness

$$K_{31} = \frac{4EI}{l} = \frac{4EI}{3} = 1,3EI$$

$$K_{34} = \frac{4EI}{l} = \frac{4EI}{5} = 0,8EI$$

$$K_{42} = \frac{3EI}{l} = \frac{4EI}{3} = EI$$

Distribution factors

-3. node:

$$r = \frac{K_{31}}{K_{31} + K_{34}} = \frac{1, \hat{3}EI}{1, \hat{3}EI + 0,8EI} = 0,625$$

$$r_{34} = 1 - 0,625 = 0,375$$

-4. node:

$$r_{43} = \frac{K_{43}}{K_{43} + K_{42}} = \frac{0,8EI}{0,8EI + 1EI} = 0,4$$

$$r_{42} = 1 - 0,4 = 0,6$$

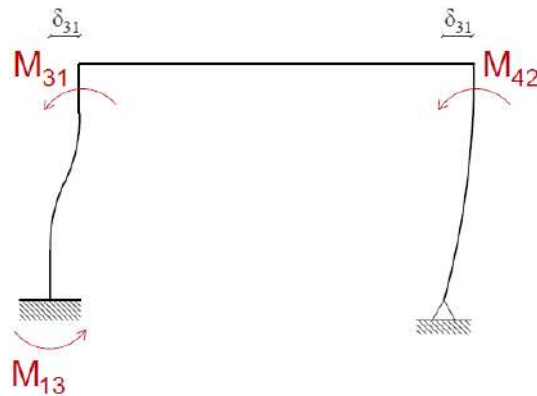
With these values we can approach the cross table of the restrained frame:

Cross Table of state 1 (restrained frame)

1, CROSS (RESTRAINED)	1	3			4			2
FEM	13		31	34		43	42	24
	-		0,625	0,375		0,444	0,556	-
	1875,0	4375,0	-1875,0	6250,0		-6250,0		
	-1367,2	-4375,0	-2734,4	-1640,6	-7070,3	-820,3		
		1571,2		1571,2	7070,3	3142,4	3928,0	
	-491,0	-1571,2	-982,0	-589,2	-294,6	-294,6		
		65,5		65,5	294,6	130,9	163,7	
	-20,5	-65,5	-40,9	-24,5	-12,3	-12,3		
					12,3	5,5	6,8	
	-3.64		-5632.28	5632.28		-4098.44	4098.44	0.00

This way, first state of our diagram is solved. The next step is to calculate the FEM of the state 2 (the sway frame).

State 2. FEM (sway frame. Step 1)



By applying the formulas seen in 5.2.2. section:

$$M_{13} = M_{31} = \delta_{13} \cdot \frac{6EI}{l_{13}^2} = 0,6 \cdot \delta_{13}EI$$

$$M_{42} = \delta_{42} \cdot \frac{3EI}{l_{42}^2} = 0,3 \cdot \delta_{42}EI$$

Even though the displacements are unknown, we can observe that, $\delta_{31} = \delta_{42}$, because of the inextensible bars condition (3-4 bar's length doesn't change). So we can express the two bar's FEM depending on a single unknown:

$$\delta_{31} = \delta_{42} = \Delta$$

Therefore:

$$M_{13} = M_{31} = 0,6 \cdot \Delta \cdot EI$$

$$M_{42} = 0,3 \cdot \Delta \cdot EI$$

But these values are not very practical to develop the second state's table. Because of this, we will rename them depending on other unknown, but with more convenient values. We will take the biggest moment we have, and rename this way:

$$M_{13} = M_{31} = 0,6 \cdot \Delta \cdot EI = 1000\alpha$$

Therefore, solving this equation for Δ :

$$M_{13} = M_{31} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha; \Delta = \frac{1000\alpha}{0, \hat{6}EI}; \Delta = \frac{1500\alpha}{EI}$$

And finally, by introducing the Δ value in the other bar's moment formula we obtain the value of M_{42} :

$$M_{42} = 0, \hat{3} \cdot \Delta \cdot EI = 0, \hat{3} \cdot \frac{1500\alpha}{EI} EI = 500\alpha$$

Now that we have the second state FEM, we can approach the second table and solve the frame. When introducing the values of the second table we are going to remove the α unknown, just for convenience. But we should remember that the whole second state's table moments depend on this unknown.

Cross Table of states 1 and 2

1, CROSS (RESTRAINED)	1	3			4			2
	13		31	34		43	42	24
FEM	-		0,625	0,375		0,444	0,556	-
	1875,0	4375,0	-1875,0	6250,0		-6250,0		
	-1367,2	-4375,0	-2734,4	-1640,6	-7070,3	-820,3		
		1571,2		1571,2	7070,3	3142,4	3928,0	
	-491,0	-1571,2	-982,0	-589,2	-294,6	-294,6		
		65,5		65,5	294,6	130,9	163,7	
	-20,5	-65,5	-40,9	-24,5	-12,3	-12,3		
					12,3	5,5	6,8	
	-3,64		-5632,28	5632,28		-4098,44	4098,44	0,00
2, CROSS (SWAY)	1	3			4			2
	13		31	34		43	42	24
FEM x α	-		0,625	0,375		0,444	0,556	-
	1000,0	1000,0	1000,0				500,0	
	-312,5	-1000,0	-625,0	-375,0	312,5	-187,5		
		-69,4		-69,4	-312,5	-138,9	-173,6	
	21,7	69,4	43,4	26,0	13,0	13,0		
		-2,9		-2,9	-13,0	-5,8	-7,2	
	0,9	2,9	1,8	1,1	0,5	0,5		
					-0,5	-0,2	-0,3	
	710,11		420,21	-420,21		-318,85	318,85	0,00

So according to the table above, the values of the moments in the ends of the bar are:

$$M_{13} = -3,64 + 710,11\alpha$$

$$M_{31} = -5632,28 + 420,21\alpha$$

$$M_{34} = 5632,28 - 420,21\alpha$$

$$M_{43} = -4098,44 - 318,85\alpha$$

$$M_{42} = 4098,44 + 318,85\alpha$$

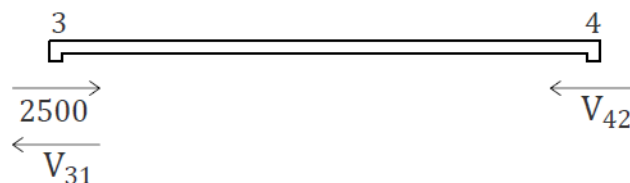
$$M_{24} = 0$$

It means that we need to find out the value of α . Let's remember what the conditions that the summary diagram have to respect to fulfil the superposition principle are (section 5.2.1):

- We need to calculate the fixed end moments to restrain the rotation during the translation
- All the actions and reactions applied in the upper-right node must add up to zero.

We already have calculated the value of the fixed end moments, but depending on α . It means that each value of α represents a displacement with restrain rotation in nodes. But among all the possible values, only one will respect the second condition. The value of α with which the addition of all the actions and reactions applied in the node 4 is null.

Now, let's analyse the displacement equilibrium equation exposed in section 5.2.4 for this frame:



$$\Sigma H = 0; 2500 = V_{31} + V_{42}; 2500 = \frac{M_{31} + M_{13}}{l_{31}} + \frac{M_{42}}{l_{42}}; \boxed{7500 = M_{31} + M_{13} + M_{42}}$$

In this equilibrium equation node 4 didn't support any external action. It implies that when solving this equation, all actions and reactions represented in summary diagram in this node add up to zero. In other words, by solving this equation we obtain the value of α that make valid the whole superposition principle summary diagram.

$$7500 = M_{31} + M_{13} + M_{42}$$

$$M_{13} = -3,64 + 710,11\alpha$$

$$M_{31} = -5632,28 + 420,21\alpha$$

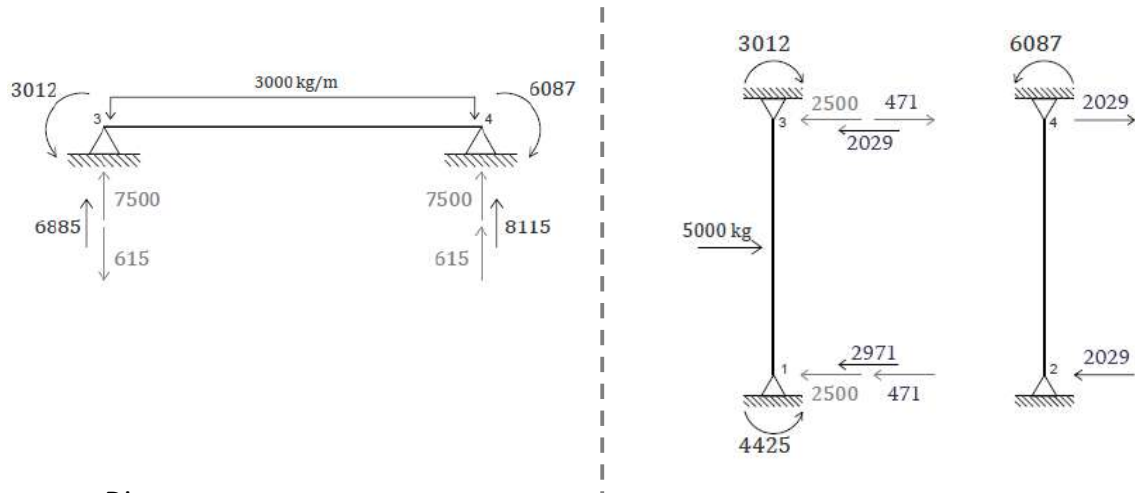
$$M_{42} = 4098,44 + 318,85\alpha$$

$$7500 = -1537,48 + 1449,17\alpha; \alpha = \frac{9037,48}{1449,17} = 6,236$$

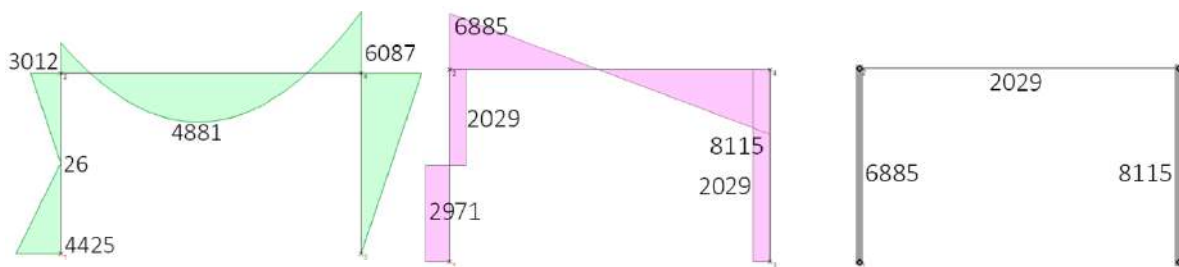
Once calculated α value we can introduce it in the table and obtain the final results for this frame

1, CROSS (RESTRAINED)		1		3		4		2	
FEM	13			31	34		43	42	24
	-			0,625	0,375		0,444	0,556	-
	1875,0	4375,0	-1875,0	6250,0		-6250,0			
	-1367,2	-4375,0	-2734,4	-1640,6	-7070,3	-820,3			
		1571,2		1571,2	7070,3	3142,4	3928,0		
	-491,0	-1571,2	-982,0	-589,2	-294,6	-294,6			
		65,5		65,5	294,6	130,9	163,7		
	-20,5	-65,5	-40,9	-24,5	-12,3	-12,3			
					12,3	5,5	6,8		
	-3,64		-5632,28	5632,28		-4098,44	4098,44	0,00	
2, CROSS (SWAY)		1		3		4		2	
FEM x α	13			31	34		43	42	24
	-			0,625	0,375		0,444	0,556	-
	1000,0	1000,0	1000,0					500,0	
	-312,5	-1000,0	-625,0	-375,0		312,5	-187,5		
		-69,4		-69,4		-312,5	-138,9	-173,6	
	21,7	69,4	43,4	26,0		13,0	13,0		
		-2,9		-2,9		-13,0	-5,8	-7,2	
	0,9	2,9	1,8	1,1		0,5	0,5		
						-0,5	-0,2	-0,3	
	710,11		420,21	-420,21		-318,85	318,85	0,00	
		1		3		4		2	
	13			31	34		43	42	24
1. CROSS		-4		-5632	5632		-4098	4098	0
2. CROSS x α=6,236		4428		2621	-2621		-1988	1988	0
FINAL RESULTS		4425		-3012	3012		-6087	6087	0

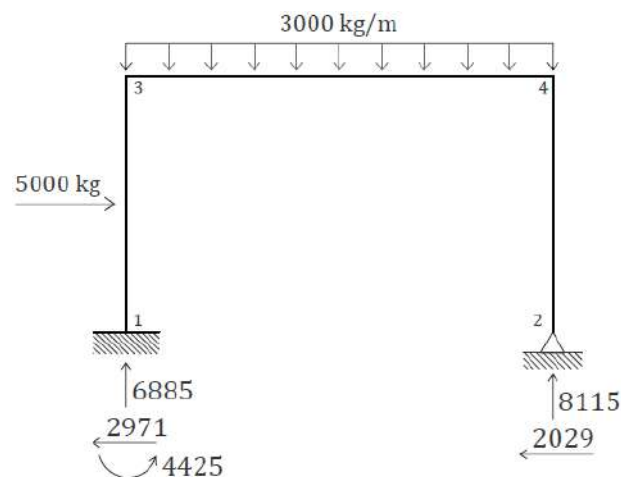
Finally, when we already have the table results, we must follow the same procedure applied in non-sway frames.



Diagrams



Reactions



$$\Sigma H = 0; 5000 = 2971 + 2029; 5000 = 5000 \quad \checkmark$$

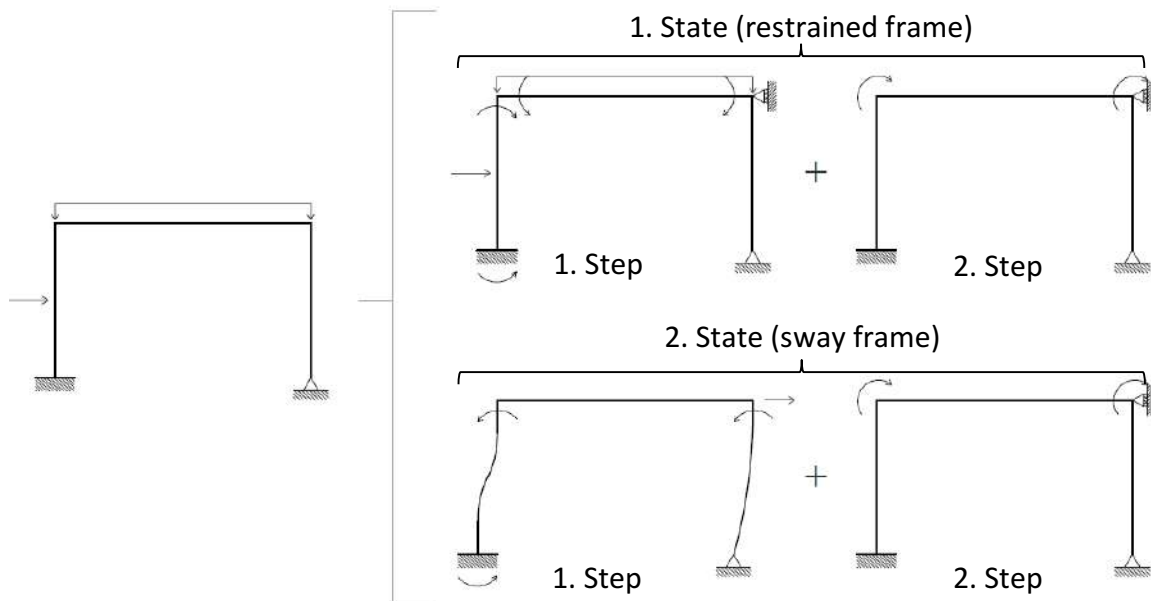
$$\Sigma V = 0; 3000 \cdot 5 = 8115 + 6885; 15000 = 15000 \quad \checkmark$$

$$\Sigma M_1 = 0; 3000 \cdot 5 \cdot 2,5 + 5000 \cdot 1,5 = 8115 \cdot 5 + 4425$$

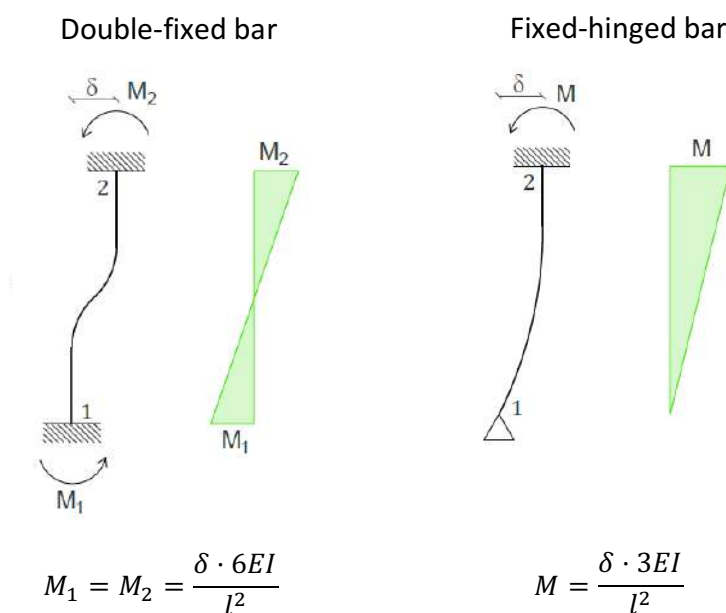
$$45000 = 40575 + 4425; 45000 = 45000 \quad \checkmark$$

5.2.6. Calculation of displacements

If we analyse the diagram of the previous frame, we observe that between the different steps for the application of the superposition principle, the only one where the nodes experience a translation is when applying the FEM of the displacements. In the first step of the second state.



Taking into account the way in which we apply the superposition principle when using Cross method, we always have one of these two situations in this step:



As the FEM of the displacement depends on α , we need to first calculate its value. In the case of the frame we are analysing, we already have obtained the value, so we can calculate the displacement.

3. node's displacement

$$\delta \cdot \frac{M \cdot l^2}{6EI} = \frac{623600 \cdot 300^2}{6 \cdot 1,35 \cdot 10^{10}} = 0,69 \text{ cm}$$
$$M = 1000\alpha = 1000 \cdot 6,236 = 6236 \text{ kg} \cdot \text{m} = 623600 \text{ kg} \cdot \text{cm}$$

} Calculated in 3-1 bar

4. node's displacement

$$\delta \cdot \frac{M \cdot l^2}{3EI} = \frac{311800 \cdot 300^2}{3 \cdot 1,35 \cdot 10^{10}} = 0,69 \text{ cm}$$
$$M = 500\alpha = 500 \cdot 6,236 = 3118 \text{ kg} \cdot \text{m} = 311800 \text{ kg} \cdot \text{cm}$$

} Calculated in 4-2 bar

By calculating the displacements of the two bars we check that they are equal, and therefore, the result fulfils the condition of inextensible bars, because when these two displacements are identical, 3-4 bar's length doesn't change.

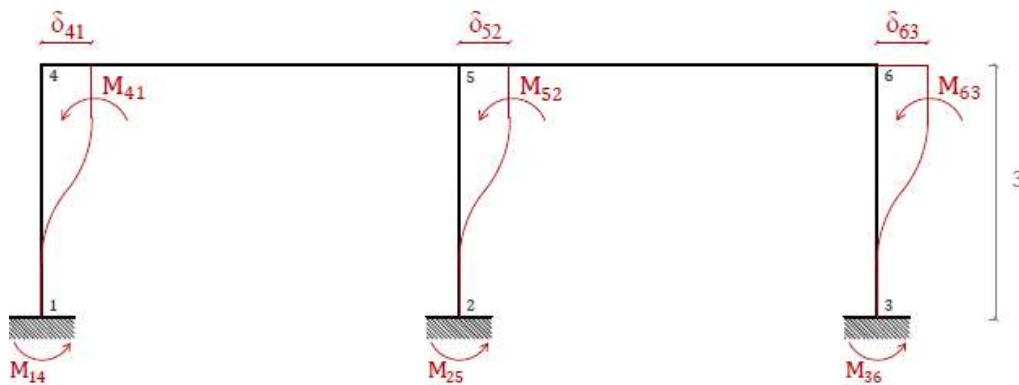
5.2.7. Translational degree and equilibrium equations. Examples

In this section we are going to study: translational degree; FEM of the displacement; and equilibrium equations in different frames, because these are the most important aspects to understand in sway structures.

We should note that the following theoretical examples are uncharged structures. That's why every equilibrium equation's moments add up to zero. In real structures it will be very important to introduce action loads in the formulas.

Orthogonal frames

1) Translational degree of freedom: 1 $\delta_{41} = \delta_{52} = \delta_{63} = \Delta$



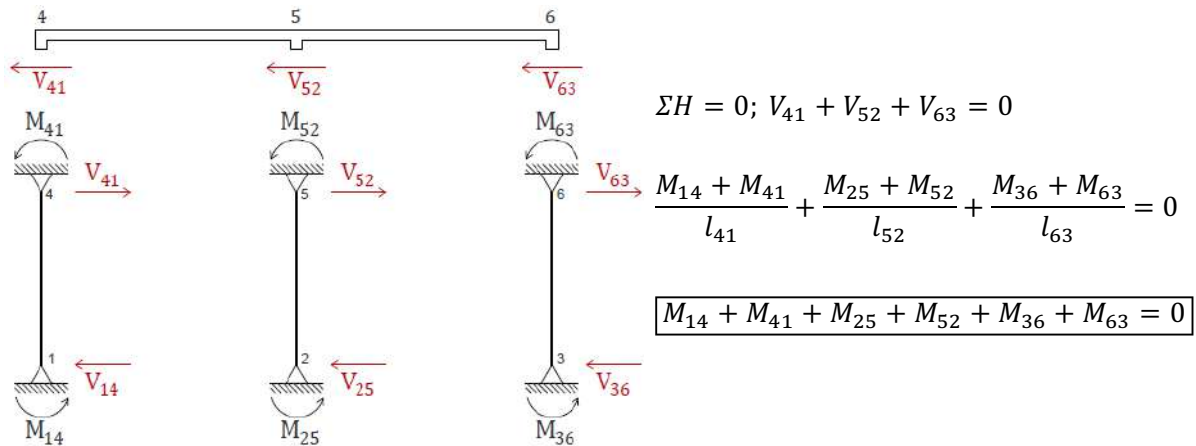
- Displacement FEM

$$M_{14} = M_{41} = \delta_{41} \cdot \frac{6EI}{l_{14}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha$$

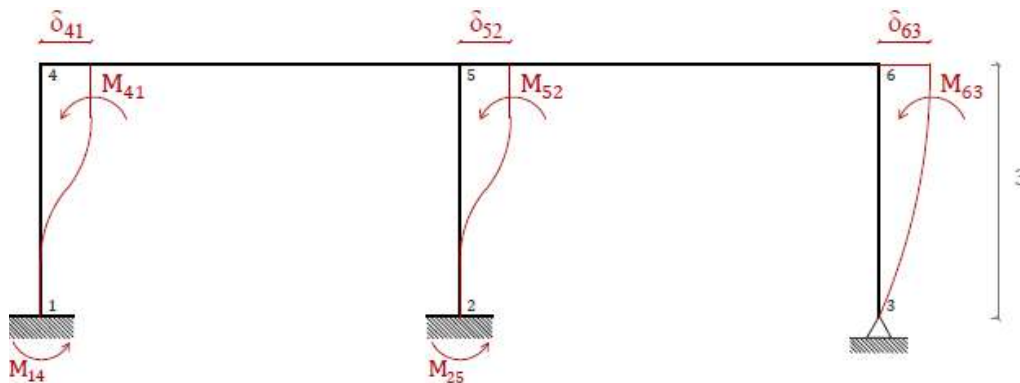
$$M_{25} = M_{52} = \delta_{52} \cdot \frac{6EI}{l_{25}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha$$

$$M_{36} = M_{63} = \delta_{63} \cdot \frac{6EI}{l_{36}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha$$

- Equilibrium equation



2) Trans. degree: 1 $\delta_{41} = \delta_{52} = \delta_{63} = \Delta$



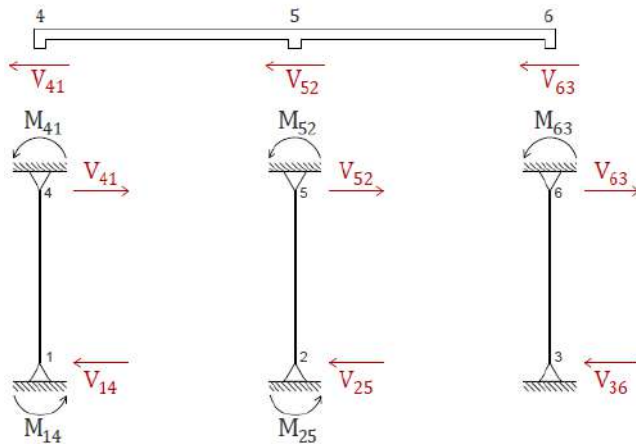
- Displacement FEM

$$M_{14} = M_{41} = \delta_{41} \cdot \frac{6EI}{l_{14}^2} = 0, \hat{6} \cdot \Delta \cdot EI = \underline{1000\alpha} \rightarrow \Delta = \frac{1500\alpha}{EI}$$

$$M_{25} = M_{52} = \delta_{52} \cdot \frac{6EI}{l_{25}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 0, \hat{6} \cdot 1500\alpha \cdot \frac{EI}{EI} = 1000\alpha$$

$$M_{63} = \delta_{63} \cdot \frac{3EI}{l_{36}^2} = 0, \hat{3} \cdot \Delta \cdot EI = 0, \hat{3} \cdot 1500\alpha \cdot \frac{EI}{EI} = 500\alpha$$

- Equilibrium equation

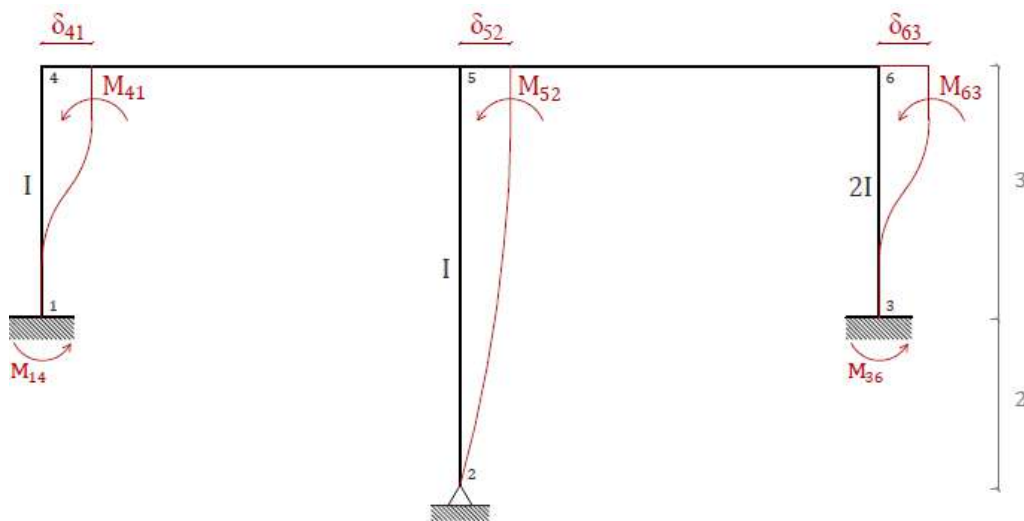


$$\Sigma H = 0; V_{41} + V_{52} + V_{63} = 0$$

$$\frac{M_{14} + M_{41}}{l_{41}} + \frac{M_{25} + M_{52}}{l_{52}} + \frac{M_{63}}{l_{63}} = 0$$

$$M_{14} + M_{41} + M_{25} + M_{52} + M_{63} = 0$$

3) Trans. degree: 1 $\delta_{41} = \delta_{52} = \delta_{63} = \Delta$



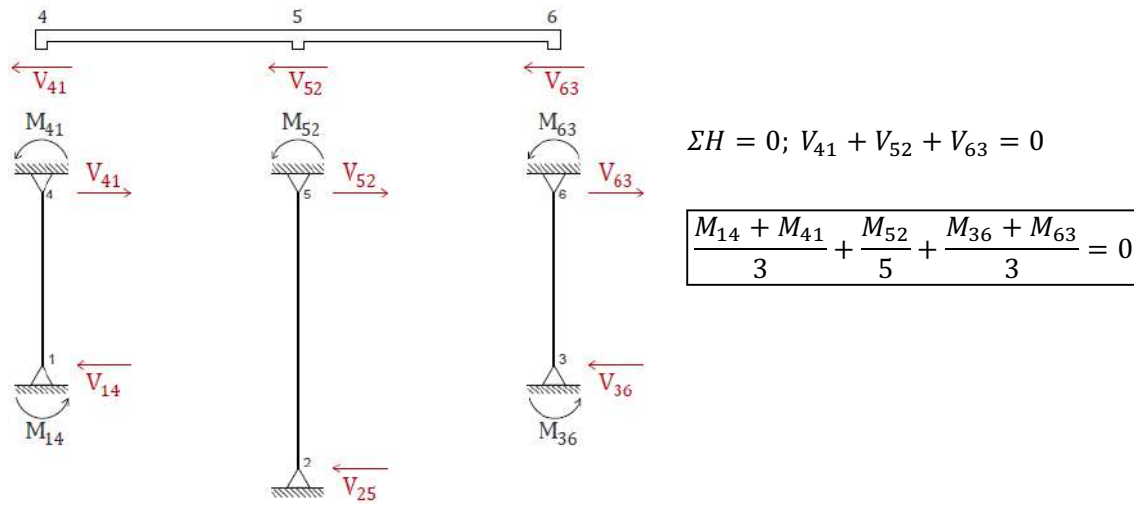
- Displacement FEM

$$M_{14} = M_{41} = \delta_{41} \cdot \frac{6EI}{l_{41}^2} = 0,6 \cdot \Delta \cdot EI = 0,6 \cdot 750\alpha \cdot \frac{EI}{EI} = 500\alpha$$

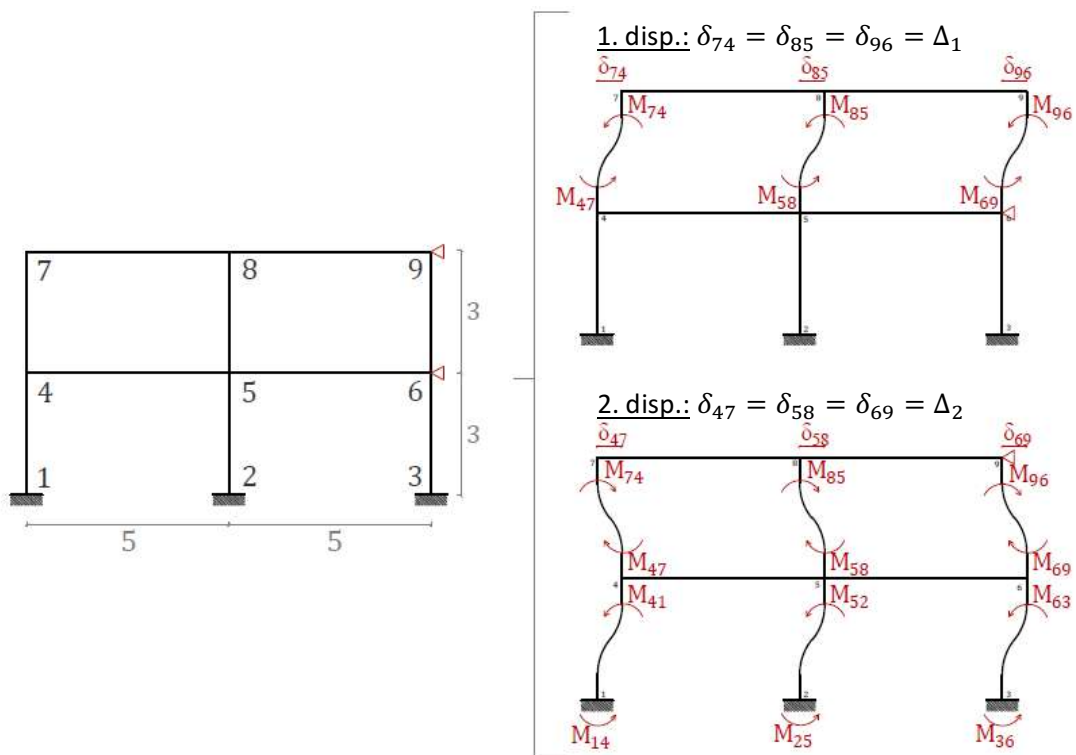
$$M_{52} = \delta_{52} \cdot \frac{3EI}{l_{25}^2} = 0,12 \cdot \Delta \cdot EI = 0,12 \cdot 750\alpha \cdot \frac{EI}{EI} = 90\alpha$$

$$M_{36} = M_{63} = \delta_{63} \cdot \frac{6E2I}{l_{36}^2} = 1,3 \cdot \Delta \cdot EI = 1000\alpha \rightarrow \Delta = \frac{750\alpha}{EI}$$

- Equilibrium equation



4) Trans. degree: 2



- Displacement 1 FEM

$$M_{47} = M_{74} = \delta_{74} \cdot \frac{6EI}{l_{47}^2} = 0, \hat{\delta} \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{58} = M_{85} = \delta_{85} \cdot \frac{6EI}{l_{58}^2} = 0, \hat{\delta} \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{69} = M_{96} = \delta_{96} \cdot \frac{6EI}{l_{69}^2} = 0, \hat{\delta} \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

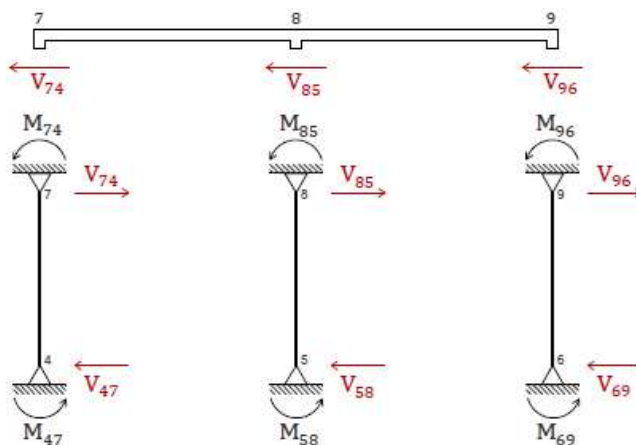
- Displacement 2 FEM

$$M_{14} = M_{41} = -M_{47} = -M_{74} = \delta_{47} \cdot \frac{6EI}{l_{47}^2} = 0, \hat{\delta} \cdot \Delta_2 \cdot EI = 1000\alpha_2$$

$$M_{25} = M_{52} = -M_{58} = -M_{85} = \delta_{58} \cdot \frac{6EI}{l_{58}^2} = 0, \hat{\delta} \cdot \Delta_2 \cdot EI = 1000\alpha_2$$

$$M_{36} = M_{63} = -M_{69} = -M_{96} = \delta_{69} \cdot \frac{6EI}{l_{69}^2} = 0, \hat{\delta} \cdot \Delta_2 \cdot EI = 1000\alpha_2$$

- Displacement 1 equilibrium equation:

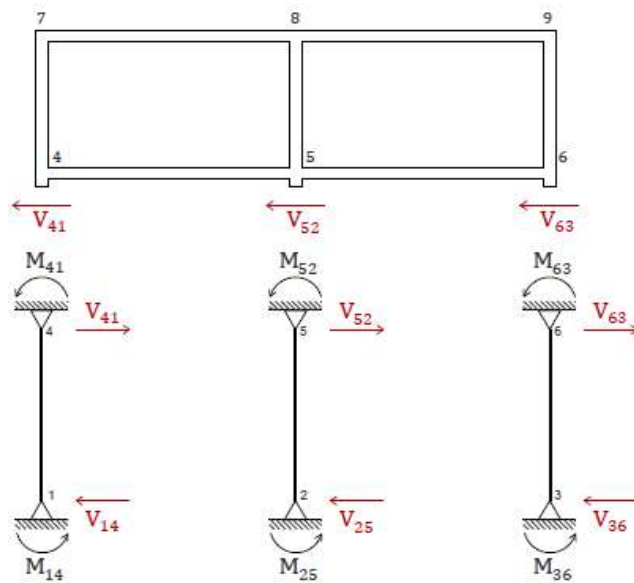


$$\Sigma H = 0; V_{74} + V_{85} + V_{96} = 0$$

$$\frac{M_{74} + M_{47}}{l_{74}} + \frac{M_{85} + M_{58}}{l_{85}} + \frac{M_{96} + M_{69}}{l_{96}} = 0$$

$$M_{74} + M_{47} + M_{85} + M_{58} + M_{96} + M_{69} = 0$$

- Displacement 2 equilibrium equation:

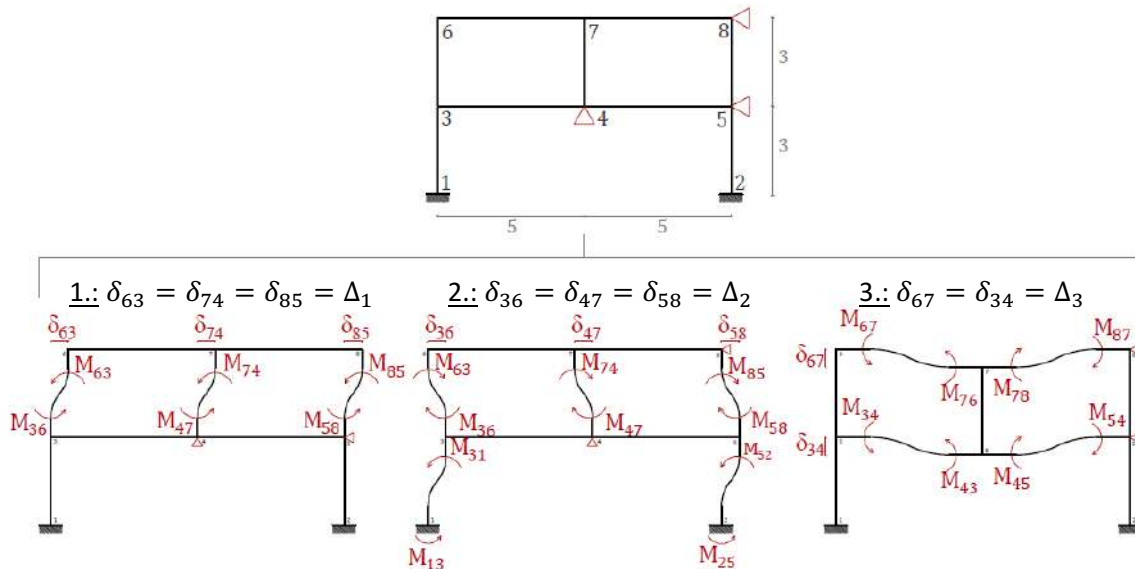


$$\Sigma H = 0; V_{41} + V_{52} + V_{63} = 0$$

$$\frac{M_{14} + M_{41}}{l_{41}} + \frac{M_{25} + M_{52}}{l_{52}} + \frac{M_{36} + M_{63}}{l_{63}} = 0$$

$$M_{14} + M_{41} + M_{25} + M_{52} + M_{36} + M_{63} = 0$$

5) Trans. degree: 3



- Displacement 1 FEM

$$M_{36} = M_{63} = \delta_{63} \cdot \frac{6EI}{l_{36}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{47} = M_{74} = \delta_{74} \cdot \frac{6EI}{l_{47}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{58} = M_{85} = \delta_{85} \cdot \frac{6EI}{l_{58}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

- Displacement 2 FEM

$$M_{13} = M_{31} = -M_{36} = -M_{63} = \delta_{36} \cdot \frac{6EI}{l_{36}^2} = 0,6 \cdot \Delta_2 \cdot EI = 1000\alpha_2$$

$$M_{47} = M_{74} = -\delta_{47} \cdot \frac{6EI}{l_{47}^2} = -0,6 \cdot \Delta_2 \cdot EI = -1000\alpha_2$$

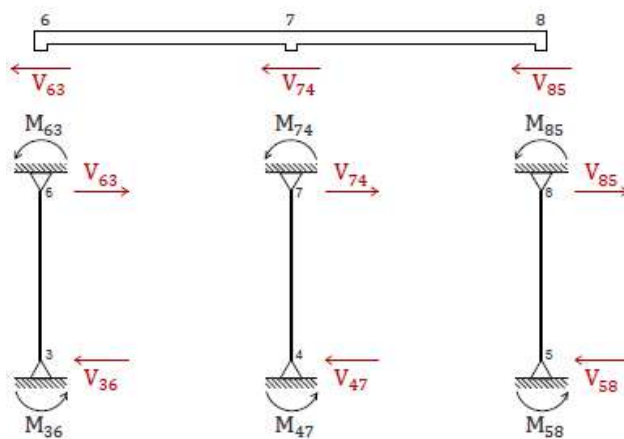
$$M_{25} = M_{52} = -M_{58} = -M_{85} = \delta_{58} \cdot \frac{6EI}{l_{58}^2} = 0,6 \cdot \Delta_2 \cdot EI = 1000\alpha_2$$

- Displacement 3 FEM

$$M_{67} = M_{76} = -M_{78} = -M_{87} = \delta_{67} \cdot \frac{6EI}{l_{67}^2} = 0,24 \cdot \Delta_3 \cdot EI = 1000\alpha_3$$

$$M_{34} = M_{43} = -M_{45} = -M_{54} = \delta_{34} \cdot \frac{6EI}{l_{34}^2} = 0,24 \cdot \Delta_3 \cdot EI = 1000\alpha_3$$

- Displacement 1 equilibrium equation:

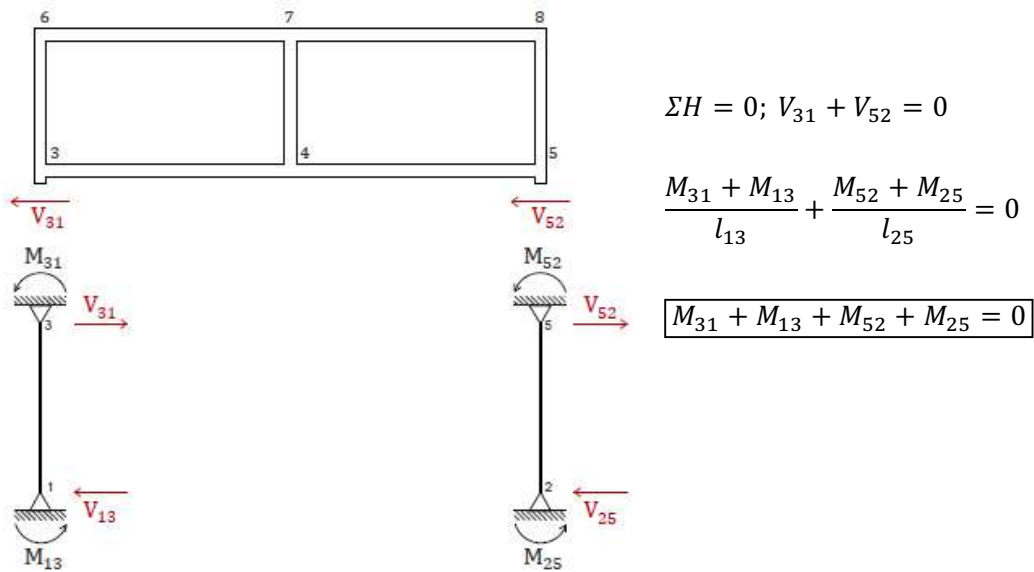


$$\Sigma H = 0; V_{63} + V_{74} + V_{85} = 0$$

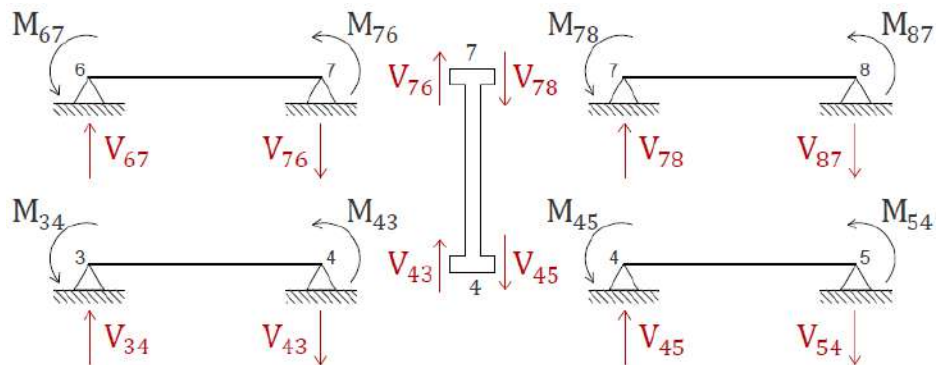
$$\frac{M_{63} + M_{36}}{l_{36}} + \frac{M_{74} + M_{47}}{l_{47}} + \frac{M_{85} + M_{58}}{l_{58}} = 0$$

$$M_{63} + M_{36} + M_{74} + M_{47} + M_{85} + M_{58} = 0$$

- Displacement 2 equilibrium equation:



- Displacement 3 equilibrium equation:

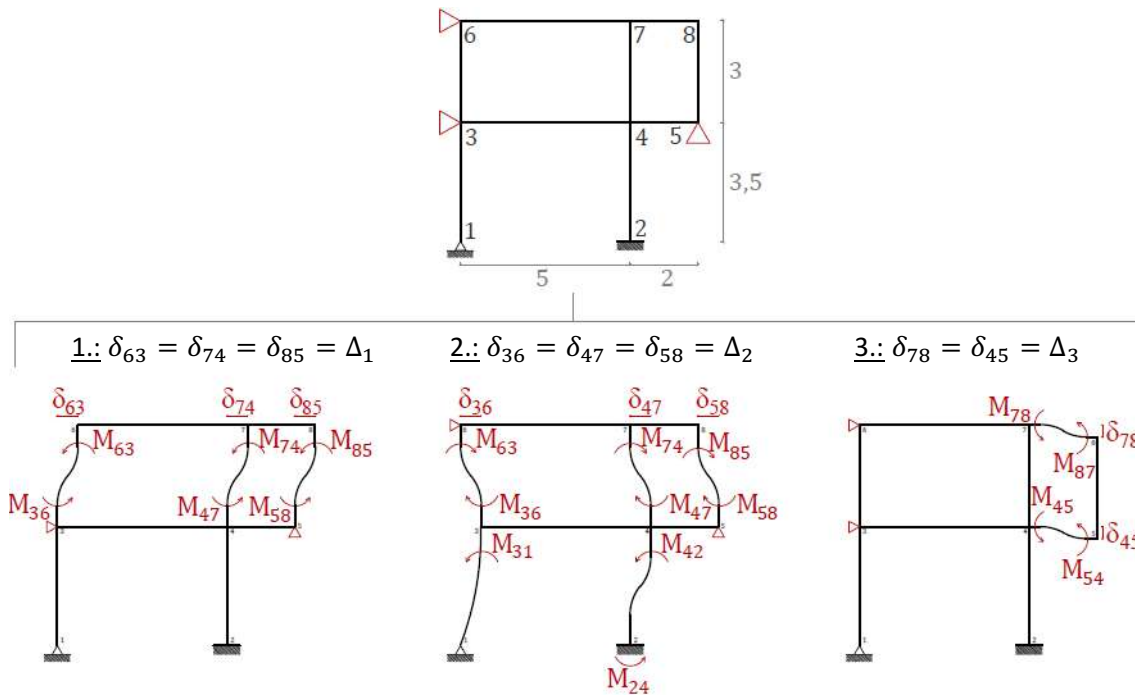


$$\Sigma V = 0; V_{76} + V_{43} = V_{78} + V_{45}$$

$$\frac{M_{67} + M_{76}}{l_{67}} + \frac{M_{34} + M_{43}}{l_{34}} = \frac{M_{78} + M_{87}}{l_{78}} + \frac{M_{45} + M_{54}}{l_{45}}$$

$$\boxed{M_{67} + M_{76} + M_{34} + M_{43} = M_{78} + M_{87} + M_{45} + M_{54}}$$

6) Trans. degree: 3



- Displacement 1 FEM

$$M_{36} = M_{63} = \delta_{63} \cdot \frac{6EI}{l_{36}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{47} = M_{74} = \delta_{74} \cdot \frac{6EI}{l_{47}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{58} = M_{85} = \delta_{85} \cdot \frac{6EI}{l_{58}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

- Displacement 2 FEM

$$M_{31} = \delta_{36} \cdot \frac{3EI}{l_{31}^2} = 0,245 \cdot \Delta_2 \cdot EI = 0,245 \cdot 1500\alpha_2 \cdot \frac{EI}{EI} = 367,5\alpha_2$$

$$M_{36} = M_{63} = -\delta_{36} \cdot \frac{6EI}{l_{36}^2} = -0,6 \cdot \Delta_2 \cdot EI = -1000\alpha_2 \rightarrow \Delta_2 = \frac{1500\alpha_2}{EI}$$

$$M_{42} = M_{24} = \delta_{47} \cdot \frac{6EI}{l_{24}^2} = 0,49 \cdot \Delta_2 \cdot EI = 0,49 \cdot 1500\alpha_2 \cdot \frac{EI}{EI} = 735\alpha_2$$

$$M_{47} = M_{74} = -\delta_{47} \cdot \frac{6EI}{l_{47}^2} = -0,6 \cdot \Delta_2 \cdot EI = -1000\alpha_2$$

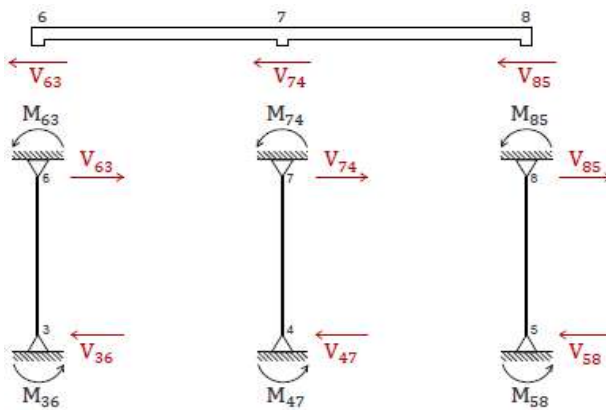
$$M_{58} = M_{85} = -\delta_{58} \cdot \frac{6EI}{l_{58}^2} = -0,6 \cdot \Delta_2 \cdot EI = -1000\alpha_2$$

- Displacement 3 FEM

$$M_{78} = M_{87} = \delta_{78} \cdot \frac{6EI}{l_{78}^2} = 1,5 \cdot \Delta_3 \cdot EI = 1000\alpha_3$$

$$M_{45} = M_{54} = \delta_{45} \cdot \frac{6EI}{l_{45}^2} = 1,5 \cdot \Delta_3 \cdot EI = 1000\alpha_3$$

- Displacement 1 equilibrium equation:

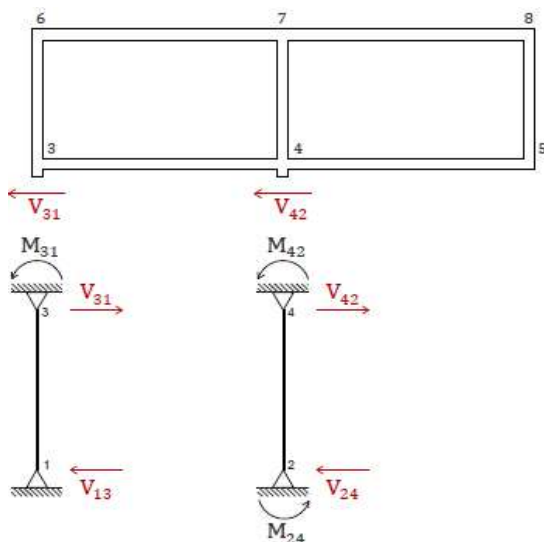


$$\Sigma H = 0; V_{63} + V_{74} + V_{85} = 0$$

$$\frac{M_{63} + M_{36}}{l_{36}} + \frac{M_{74} + M_{47}}{l_{47}} + \frac{M_{85} + M_{58}}{l_{58}} = 0$$

$$\boxed{M_{63} + M_{36} + M_{74} + M_{47} + M_{85} + M_{58} = 0}$$

- Displacement 2 equilibrium equation:

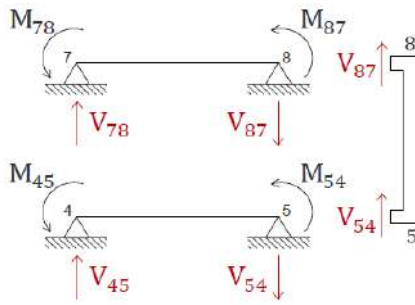


$$\Sigma H = 0; V_{31} + V_{42} = 0$$

$$\frac{M_{31}}{l_{13}} + \frac{M_{42} + M_{24}}{l_{24}} = 0$$

$$\boxed{M_{31} + M_{42} + M_{24} = 0}$$

- Displacement 3 equilibrium equation:



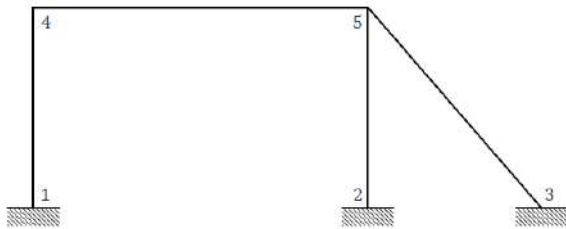
$$\Sigma V = 0; V_{87} + V_{54} = 0$$

$$\frac{M_{87} + M_{78}}{l_{78}} + \frac{M_{45} + M_{54}}{l_{45}} = 0$$

$$\boxed{M_{87} + M_{78} + M_{45} + M_{54} = 0}$$

Inclined bar's frames

1) Trans. degree: 0. Restrained frame.



$$d = 2n - b - r = 10 - 6 - 4 = 0$$

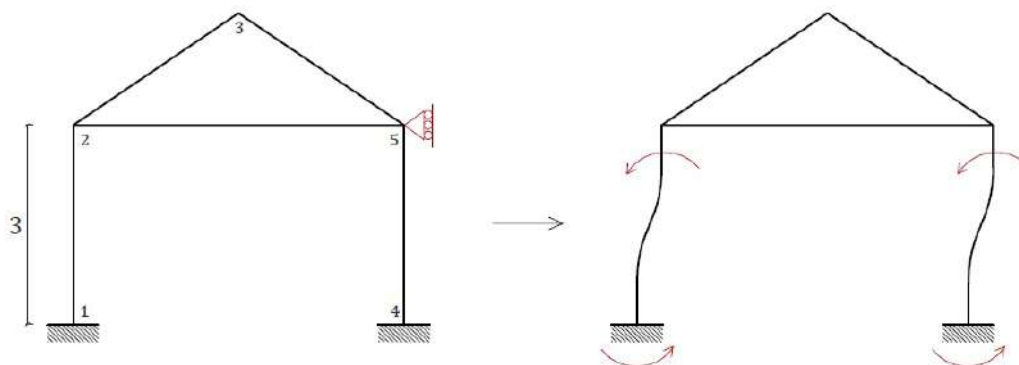
$$n = 5$$

$$r = 4$$

$$b = 6$$

The following two frames are symmetrical. It means that if we apply symmetrical loading to them, they will not have displacements, whatever their translational degree is.

2) Trans. degree: 1



$$d = 2n - b - r = 10 - 5 - 4 = 1$$

$$n = 5$$

$$r = 5$$

$$b = 4$$

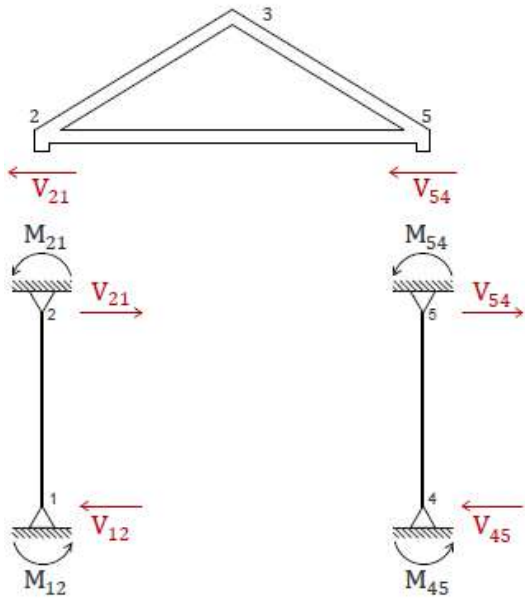
- Displacement FEM

$$\delta_{21} = \delta_{45} = \Delta$$

$$M_{12} = M_{21} = \delta_{21} \cdot \frac{6EI}{l_{21}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha$$

$$M_{45} = M_{54} = \delta_{45} \cdot \frac{6EI}{l_{45}^2} = 0, \hat{6} \cdot \Delta \cdot EI = 1000\alpha$$

- Displacement Equilibrium equation:

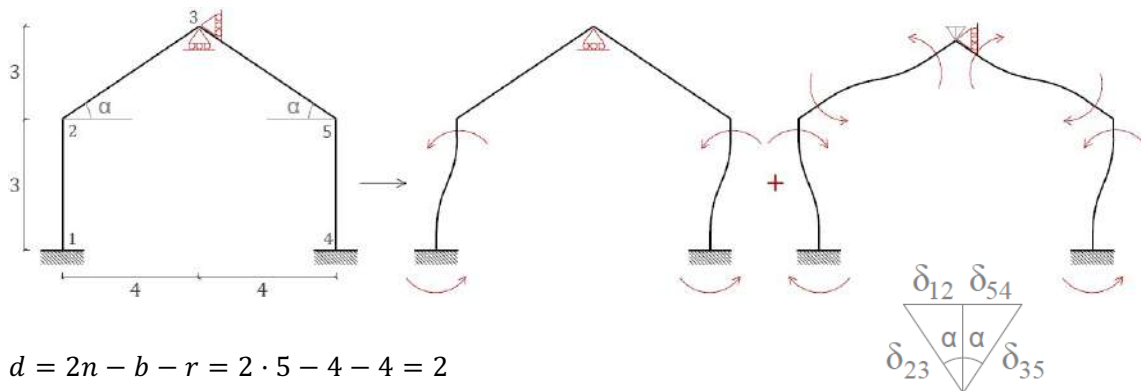


$$\Sigma H = 0; V_{21} + V_{54} = 0$$

$$\frac{M_{12} + M_{21}}{l_{12}} + \frac{M_{45} + M_{54}}{l_{45}} + = 0$$

$$M_{12} + M_{21} + M_{45} + M_{54} = 0$$

3) Trans. degree: 2



$$d = 2n - b - r = 2 \cdot 5 - 4 - 4 = 2$$

- Displacement 1 FEM

$$\delta_{21} = \delta_{45} = \Delta_1$$

$$M_{12} = M_{21} = \delta_{21} \cdot \frac{6EI}{l_{21}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{45} = M_{54} = \delta_{45} \cdot \frac{6EI}{l_{45}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

- Displacement 2 FEM

$$\sin \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{4}{5} = 0,8$$

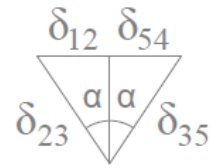
$$\delta_{23} = \delta_{35} = \Delta_2 ; \delta_{12} = \delta_{54} = \delta_{23} \cdot \sin \alpha = 0,6 \Delta_2$$

$$M_{23} = M_{32} = \delta_{23} \frac{6EI}{l_{23}^2} = \Delta_2 \cdot \frac{6EI}{5^2} = 0,24 \Delta_2 \cdot EI = 600 \alpha_2$$

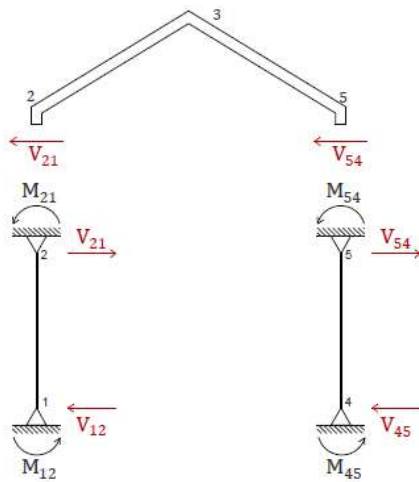
$$M_{35} = M_{53} = -\delta_{35} \frac{6EI}{l_{35}^2} = -\Delta_2 \cdot \frac{6EI}{5^2} = -0,24 \Delta_2 \cdot EI = -600 \alpha_2$$

$$M_{54} = M_{45} = \delta_{54} \frac{6EI}{l_{45}^2} = 0,6 \Delta_2 \cdot \frac{6EI}{3^2} = 0,4 \Delta_2 \cdot EI = 1000 \alpha_2 \rightarrow \Delta_2 = \frac{2500 \alpha_2}{EI}$$

$$M_{12} = M_{21} = -\delta_{12} \frac{6EI}{l_{12}^2} = -0,6 \Delta_2 \cdot \frac{6EI}{3^2} = -0,4 \Delta_2 \cdot EI = -1000 \alpha_2$$



- Displacement 1 equilibrium equation

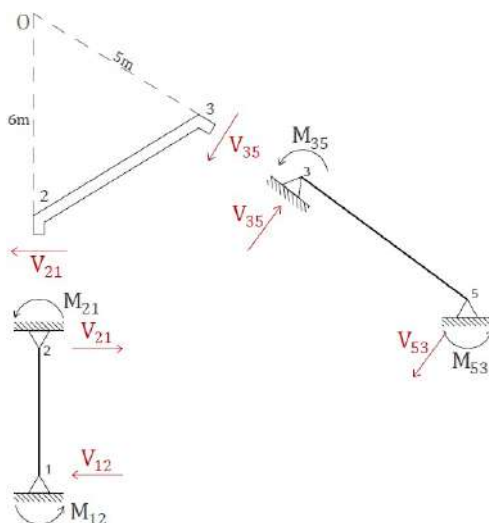


$$\Sigma H = 0; V_{21} + V_{54} = 0$$

$$\frac{M_{12} + M_{21}}{l_{12}} + \frac{M_{45} + M_{54}}{l_{45}} = 0$$

$$\boxed{M_{12} + M_{21} + M_{45} + M_{54} = 0}$$

- Displacement 2 equilibrium equation



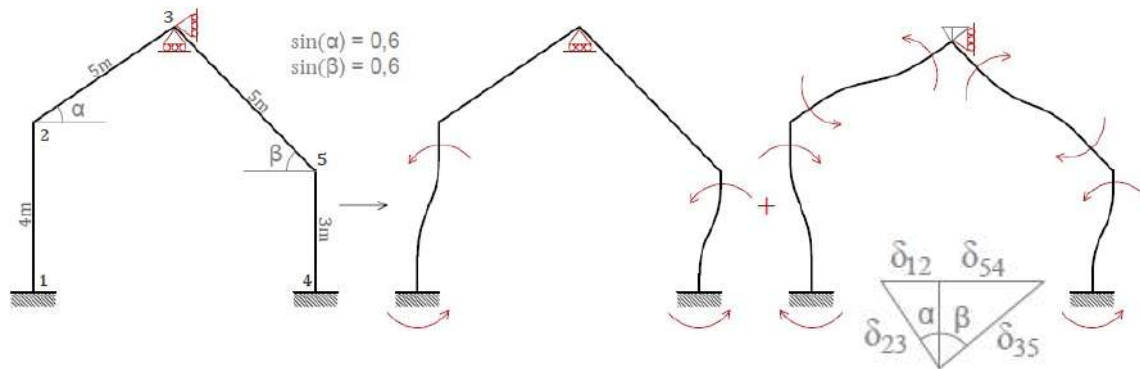
$$\Sigma M_O = 0$$

$$-6V_{21} - 5V_{35} - M_{21} - M_{35} = 0$$

$$-6 \frac{M_{12} + M_{21}}{l_{12}} - 5 \frac{M_{35} + M_{53}}{l_{35}} - M_{21} - M_{35} = 0$$

$$\boxed{2M_{12} + 3M_{21} + 2M_{35} + M_{53} = 0}$$

4) Trans. degree: 2



$$d = 2n - b - r = 10 - 4 - 4 = 2$$

- Displacement 1 FEM

$$\delta_{21} = \delta_{45} = \Delta_1$$

$$M_{12} = M_{21} = \delta_{21} \cdot \frac{6EI}{l_{21}^2} = 0,375 \cdot \Delta_1 \cdot EI = 562,5\alpha_1$$

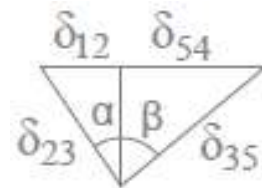
$$M_{45} = M_{54} = \delta_{45} \cdot \frac{6EI}{l_{45}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1 \rightarrow \Delta_2 = \frac{1500\alpha_1}{EI}$$

- Displacement 2 FEM

$$\sin \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{4}{5} = 0,8 ; \sin \beta = \frac{4}{5} = 0,8 ; \cos \beta = \frac{3}{5} = 0,6$$

$$\delta_{35} = \Delta_2 ; \delta_{23} = \Delta_2 \cdot \cos \beta / \cos \alpha = 0,75\Delta_2 ;$$

$$\delta_{12} = \delta_{23} \cdot \sin \alpha = 0,45\Delta_2 ; \delta_{54} = \delta_{35} \cdot \sin \beta = 0,8\Delta_2$$



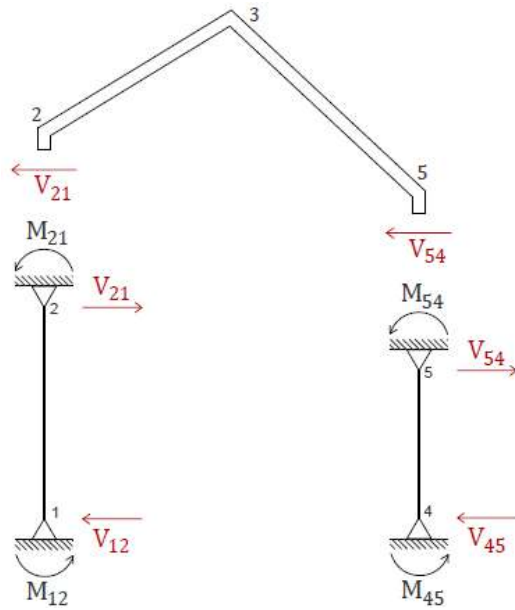
$$M_{23} = M_{32} = \delta_{23} \frac{6EI}{l_{23}^2} = 0,75\Delta_2 \cdot \frac{6EI}{5^2} = 0,18 \Delta_2 \cdot EI = 337,5 \alpha_2$$

$$M_{35} = M_{53} = -\delta_{35} \frac{6EI}{l_{35}^2} = \Delta_2 \cdot \frac{6EI}{5^2} = -0,24 \Delta_2 \cdot EI = -450\alpha_2$$

$$M_{54} = M_{45} = \delta_{54} \frac{6EI}{l_{45}^2} = 0,8 \Delta_2 \cdot \frac{6EI}{3^2} = 0,53 \Delta_2 \cdot EI = 1000\alpha_2 \rightarrow \Delta_2 = \frac{1875\alpha_2}{EI}$$

$$M_{12} = M_{21} = -\delta_{12} \frac{6EI}{l_{12}^2} = -0,45 \Delta_2 \cdot \frac{6EI}{4^2} = -0,16875 \Delta_2 \cdot EI = -316,41 \alpha_2$$

- Displacement 1 equilibrium equation

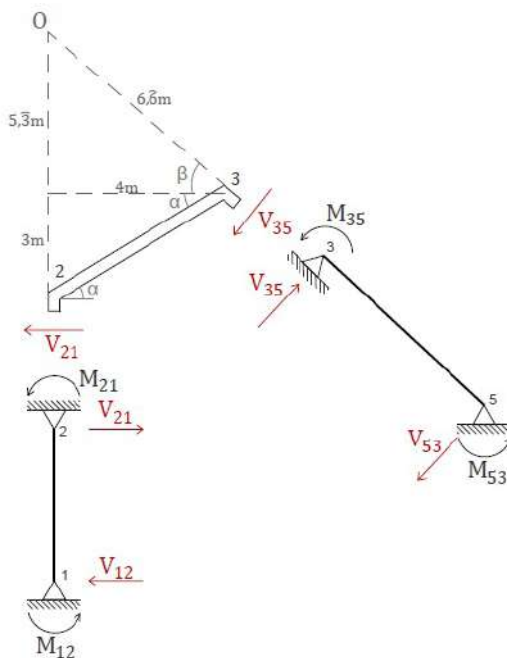


$$\Sigma H = 0; V_{21} + V_{54} = 0$$

$$\frac{M_{12} + M_{21}}{l_{12}} + \frac{M_{45} + M_{54}}{l_{45}} = 0$$

$$\frac{M_{12} + M_{21}}{4} + \frac{M_{45} + M_{54}}{3} = 0$$

- Displacement 2 equilibrium equation



$$\Sigma M_O = 0$$

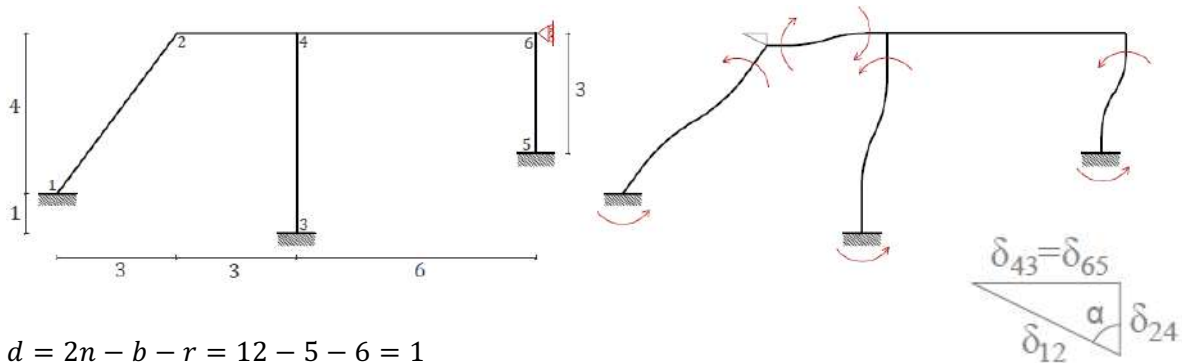
$$-8,3V_{21} - 6,6V_{35} - M_{21} - M_{35} = 0$$

$$-8,3\frac{M_{12} + M_{21}}{l_{12}} - 6,6\frac{M_{35} + M_{53}}{l_{35}} - M_{21} - M_{35} = 0$$

$$-8,3\frac{M_{12} + M_{21}}{4} - 6,6\frac{M_{35} + M_{53}}{5} - M_{21} - M_{35} = 0$$

$$2,083M_{12} + 3,083M_{21} + 2,3M_{35} + 1,3M_{53} = 0$$

5) Trans. degree: 1



$$d = 2n - b - r = 12 - 5 - 6 = 1$$

- Displacement FEM

$$\sin \alpha = \frac{4}{5} = 0,8 ; \cos \alpha = \frac{3}{5} = 0,6$$

$$\delta_{21} = \Delta ; \delta_{43} = \delta_{65} = \delta_{21} \cdot \sin \alpha = 0,8\Delta ; \delta_{24} = \delta_{21} \cdot \cos \alpha = 0,6\Delta ;$$

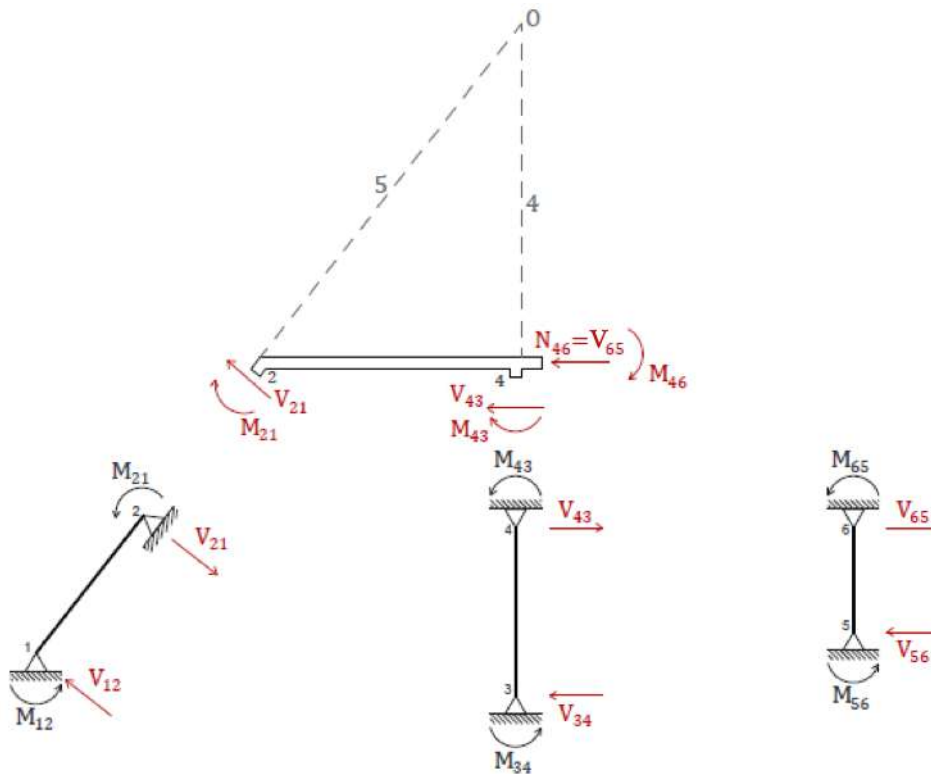
$$M_{12} = M_{21} = \delta_{21} \cdot \frac{6EI}{l_{21}^2} = 0,24 \cdot \Delta \cdot EI = 600\alpha$$

$$M_{24} = M_{42} = -\delta_{45} \cdot \frac{6EI}{l_{42}^2} = -0,4 \cdot \Delta \cdot EI = -1000\alpha \rightarrow \Delta_2 = \frac{2500\alpha_2}{EI}$$

$$M_{34} = M_{43} = \delta_{34} \cdot \frac{6EI}{l_{34}^2} = 0,192 \cdot \Delta \cdot EI = 480\alpha$$

$$M_{56} = M_{65} = \delta_{56} \cdot \frac{6EI}{l_{56}^2} = 0,192 \cdot \Delta \cdot EI = 480\alpha$$

- Displacement equilibrium equation

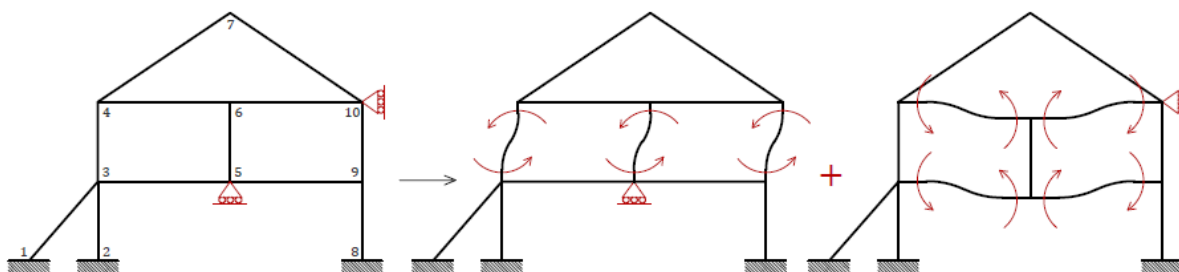


$$\Sigma M_O = 0 ; M_{46} + M_{21} + M_{43} + 5 \cdot V_{21} + 4 \cdot V_{43} + 4 \cdot V_{65} = 0$$

$$M_{46} + M_{21} + M_{43} + 5 \frac{M_{21} + M_{12}}{5} + 4 \frac{M_{43} + M_{34}}{5} + 4 \frac{M_{65} + M_{56}}{3} = 0$$

$$M_{46} + 2 M_{21} + M_{12} + 1,8 M_{43} + 0,8 M_{34} + 1, \hat{3} M_{65} + 1, \hat{3} M_{56} = 0$$

6) Trans. degree: 2



$$d = 2n - b - r = 20 - 12 - 6 = 2$$

- Displacement 1 FEM

$$\delta_{34} = \delta_{56} = \delta_{910} = \Delta_1$$

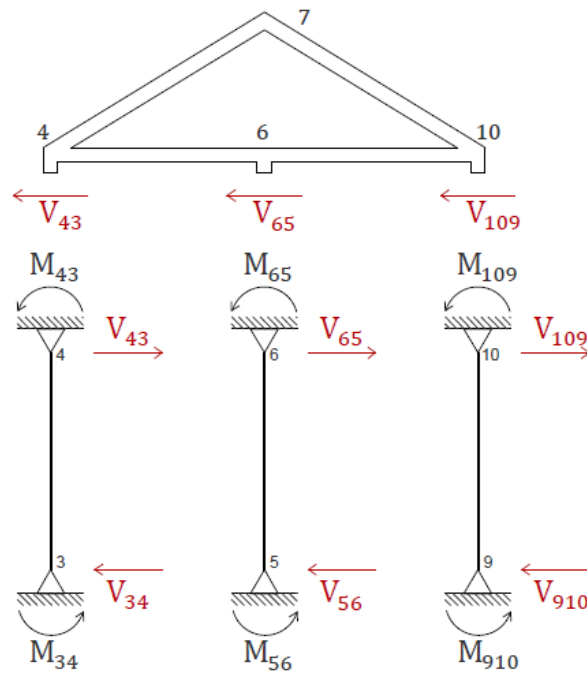
$$M_{34} = M_{43} = M_{56} = M_{65} = M_{910} = M_{109} = 1000\alpha_1$$

- Displacement 2 FEM

$$\delta_{34} = \delta_{56} = \delta_{910} = \delta_{910} = \Delta_2$$

$$M_{35} = M_{53} = M_{46} = M_{64} = -M_{59} = -M_{95} = -M_{610} = -M_{106} = 1000\alpha_2$$

- Displacement 1 equilibrium equation

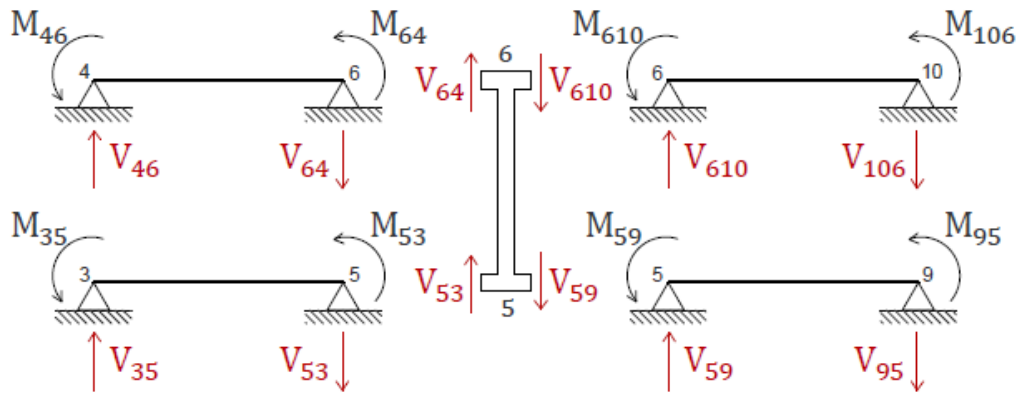


$$\Sigma H = 0; V_{43} + V_{65} + V_{109} = 0$$

$$\frac{M_{43} + M_{34}}{l_{34}} + \frac{M_{65} + M_{56}}{l_{56}} + \frac{M_{109} + M_{910}}{l_{910}} = 0$$

$$\boxed{M_{43} + M_{34} + M_{65} + M_{56} + M_{109} + M_{910} = 0}$$

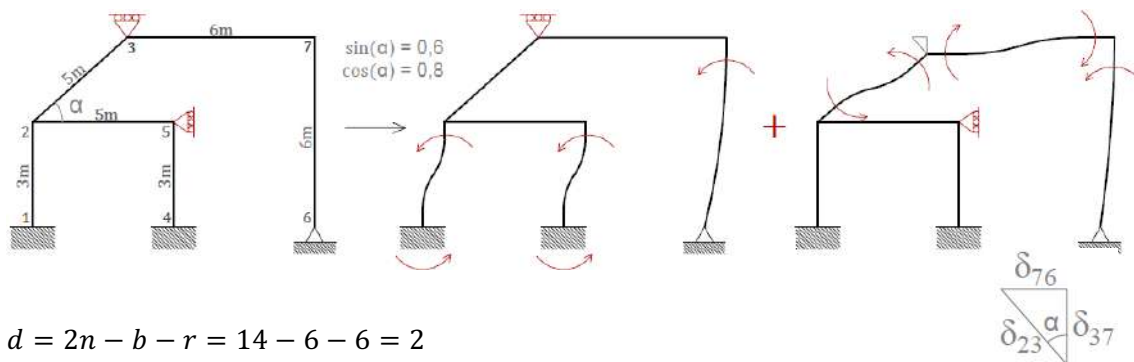
- Displacement 2 equilibrium equation



$$\sum V = 0; V_{64} + V_{53} = V_{610} + V_{59}$$

$$M_{46} + M_{64} + M_{35} + M_{53} = M_{610} + M_{106} + M_{59} + M_{95}$$

- 7) Trans. degree: 2



$$d = 2n - b - r = 14 - 6 - 6 = 2$$

- Displacement 1 FEM

$$\delta_{21} = \delta_{54} = \delta_{76} = \Delta_1$$

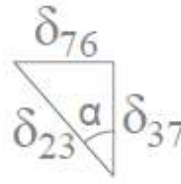
$$M_{21} = M_{12} = \delta_{21} \frac{6EI}{l_{21}^2} = \Delta_1 \cdot \frac{6EI}{3^2} = 0,6 \Delta_1 EI = 1000 \alpha_1 \rightarrow \Delta_1 = \frac{1500 \alpha_1}{EI}$$

$$M_{54} = M_{45} = \delta_{54} \frac{6EI}{l_{54}^2} = \Delta_1 \cdot \frac{6EI}{3^2} = 0,6 \Delta_1 EI = 1000 \alpha_1$$

$$M_{76} = \delta_{76} \frac{3EI}{l_{76}^2} = \Delta_1 \cdot \frac{3EI}{6^2} = 0,083 \Delta_1 EI = 125 \alpha_1$$

- Displacement 2 FEM

$$\sin \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{4}{5} = 0,8$$



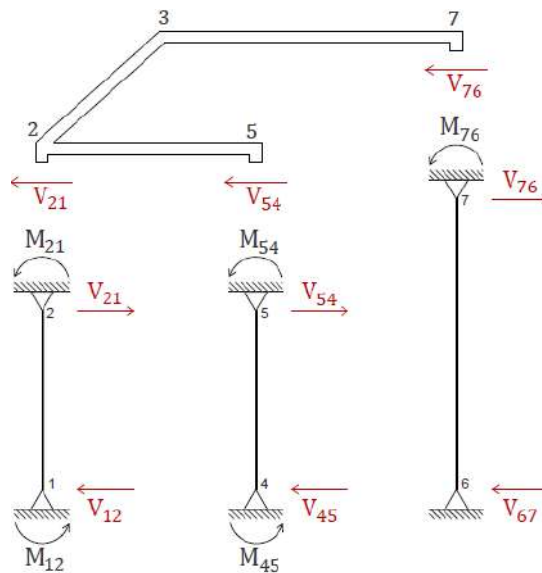
$$\delta_{23} = \Delta_2 ; \delta_{76} = 0,6\Delta_2 ; \delta_{37} = 0,8\Delta_2$$

$$M_{23} = M_{32} = \delta_{23} \frac{6EI}{l_{23}^2} = \Delta_2 \cdot \frac{6EI}{5^2} = 0,24 \Delta_2 EI = 1000 \alpha_2 \rightarrow \Delta_2 = \frac{4166,6\alpha_2}{EI}$$

$$M_{37} = M_{73} = -\delta_{37} \frac{6EI}{l_{37}^2} = -0,8\Delta_2 \cdot \frac{6EI}{6^2} = -0,1\hat{3} \Delta_2 EI = -555, \hat{5} \alpha_2$$

$$M_{76} = \delta_{76} \frac{3EI}{l_{76}^2} = 0,6\Delta_2 \cdot \frac{3EI}{6^2} = 0,1 \Delta_2 EI = 416, \hat{6} \alpha_2$$

- Displacement 1 equilibrium equation



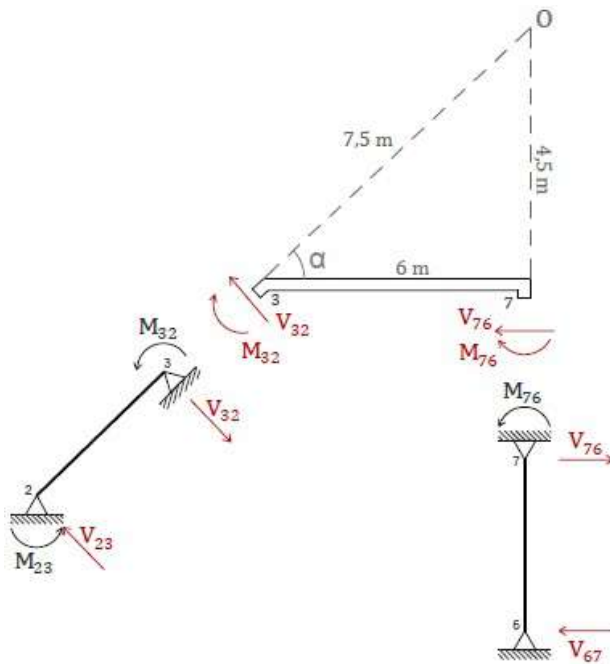
$$\Sigma H = 0 ; V_{21} + V_{54} + V_{76} = 0$$

$$\frac{M_{12} + M_{21}}{l_{12}} + \frac{M_{45} + M_{54}}{l_{45}} + \frac{M_{76}}{l_{76}} = 0$$

$$\frac{M_{12} + M_{21}}{3} + \frac{M_{45} + M_{54}}{3} + \frac{M_{76}}{6} = 0$$

$$\boxed{2M_{12} + 2M_{21} + 2M_{45} + 2M_{54} + M_{76} = 0}$$

- Displacement 2 equilibrium equation



$$\Sigma M_O = 0$$

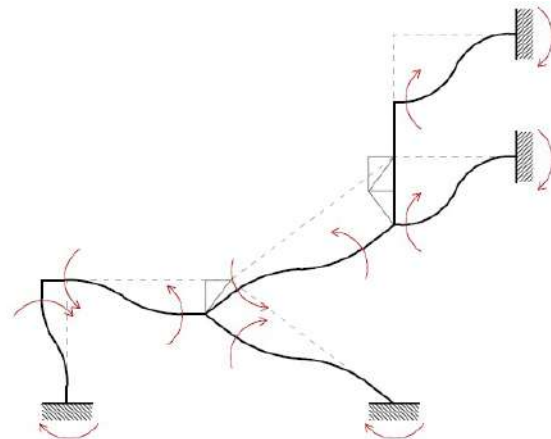
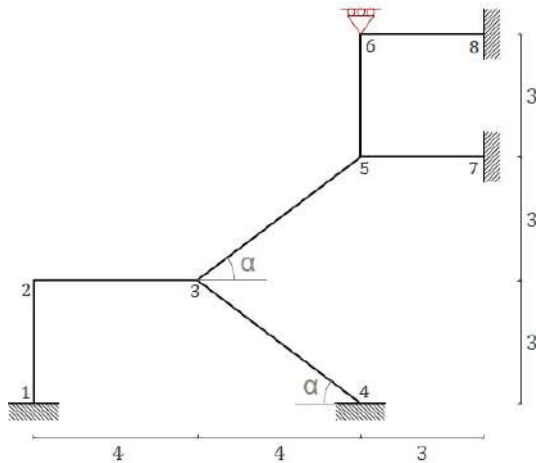
$$-7,5V_{32} - 4,5V_{76} - M_{32} - M_{76} = 0$$

$$-7,5 \frac{M_{23} + M_{32}}{l_{23}} - 4,5 \frac{M_{76}}{l_{76}} - M_{32} - M_{76} = 0$$

$$-7,5 \frac{M_{23} + M_{32}}{5} - 4,5 \frac{M_{76}}{6} - M_{32} - M_{76} = 0$$

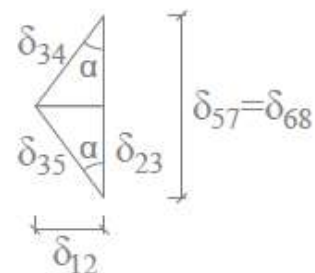
$$1,5M_{23} + 2,5M_{32} + 1,75M_{76} = 0$$

8) Trans. degree: 1



$$d = 2n - b - r = 16 - 7 - 8 = 1$$

$$n = 8 \quad r = 7 \quad b = 8$$



- Displacement FEM

$$\sin \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{4}{5} = 0,8$$

$$\delta_{34} = \delta_{35} = \Delta ; \delta_{12} = 0,6\Delta ; \delta_{23} = 0,8\Delta ; \delta_{57} = \delta_{68} = 1,6\Delta$$

$$M_{57} = M_{75} = -\delta_{57} \frac{6EI}{l_{57}^2} = -1,6\Delta \frac{6EI}{3^2} = -1,06 \Delta EI = -1000 \alpha \rightarrow \Delta = \frac{937,5\alpha}{EI}$$

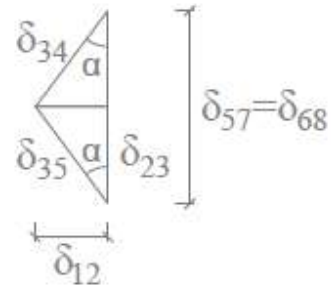
$$M_{68} = M_{86} = -\delta_{68} \frac{6EI}{l_{68}^2} = -1,6\Delta \frac{6EI}{3^2} = -1,06 \Delta EI = -1000 \alpha$$

$$M_{23} = M_{32} = \delta_{23} \frac{6EI}{l_{23}^2} = 0,8\Delta \frac{6EI}{4^2} = 0,3 \Delta EI = 281,25 \alpha$$

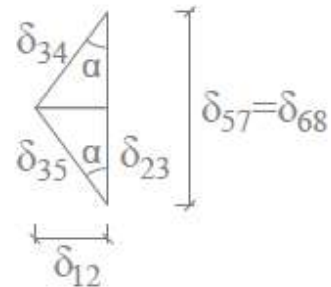
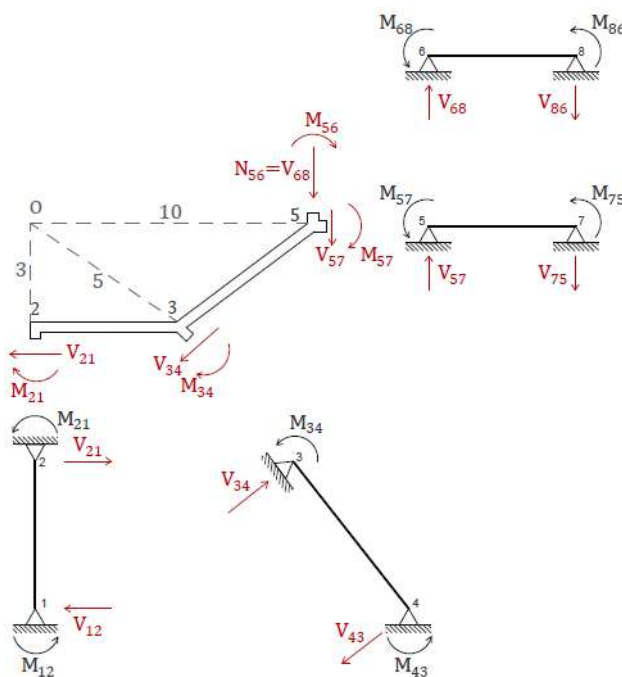
$$M_{35} = M_{53} = \delta_{35} \frac{6EI}{l_{35}^2} = \Delta \frac{6EI}{5^2} = 0,24 \Delta EI = 224 \alpha$$

$$M_{34} = M_{43} = -\delta_{34} \frac{6EI}{l_{34}^2} = -\Delta \frac{6EI}{5^2} = -0,24 \Delta EI = -224 \alpha$$

$$M_{12} = M_{21} = -\delta_{12} \frac{6EI}{l_{12}^2} = -0,6\Delta \frac{6EI}{3^2} = -0,4 \Delta EI = -375 \alpha$$



- Displacement equilibrium equation

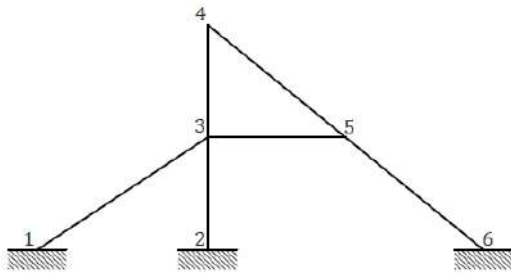


$$\Sigma M_O = 0 ; M_{21} + M_{34} + M_{57} + M_{56} + 3 \cdot V_{21} + 5 \cdot V_{34} + 10 \cdot (V_{57} + V_{68}) = 0$$

$$M_{21} + M_{34} + M_{57} + M_{56} + 3 \frac{M_{21} + M_{12}}{3} + 5 \frac{M_{43} + M_{34}}{5} + 10 \left(\frac{M_{57} + M_{75}}{3} + \frac{M_{68} + M_{86}}{3} \right) = 0$$

$$2 M_{21} + M_{12} + 2 M_{34} + M_{43} + 4, \hat{3} M_{57} + 3, \hat{3} M_{75} + M_{56} + 3, \hat{3} M_{68} + 3, \hat{3} M_{86} = 0$$

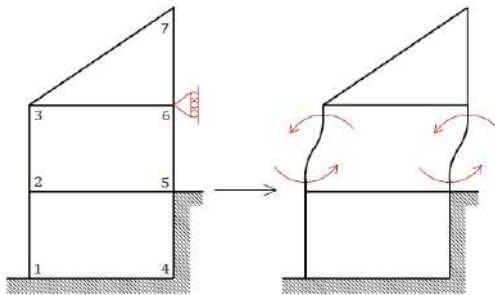
9) No displacements. Restrained frame



$$d = 2n - b - r = 12 - 6 - 6 = 0$$

$$n = 6 \quad r = 6 \quad b = 6$$

10) Trans. degree: 1



$$d = 2n - b - r = 12 - 7 - 4 = 1$$

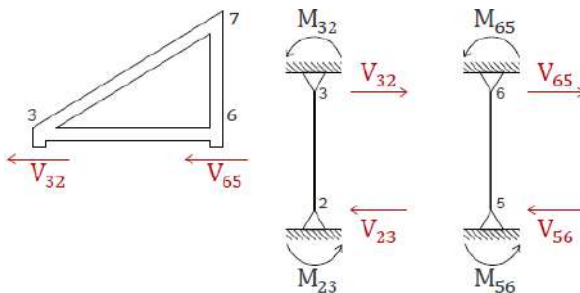
$$n = 6 \quad r = 7 \quad b = 4$$

- Displacement FEM

$$\delta_{23} = \delta_{56} = \Delta; \quad l_{23} = l_{56} = l$$

$$M_{23} = M_{32} = M_{56} = M_{65} = \Delta \cdot \frac{6EI}{l^2} = 1000\alpha$$

- Displacement equilibrium equation



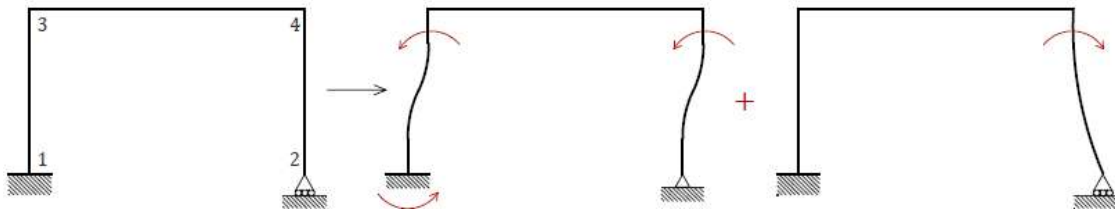
$$\Sigma H = 0; \quad V_{32} + V_{65} = 0$$

$$\frac{M_{32} + M_{23}}{l_{23}} + \frac{M_{56} + M_{65}}{l_{56}} = 0$$

$$\boxed{M_{32} + M_{23} + M_{56} + M_{65} = 0}$$

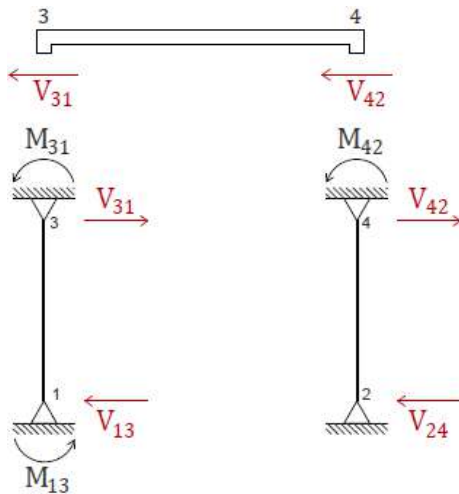
Roller supports. Equilibrium equations

1) Trans. degree: 2



$$d = 2n - b - r = 8 - 3 - 3 = 2$$

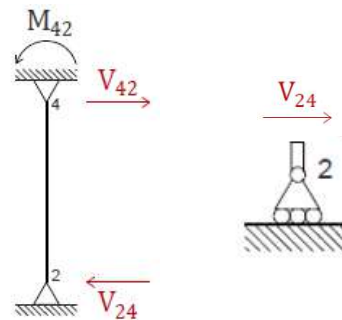
Displacement 1 E.E.



$$\sum H = 0; V_{31} + V_{42} = 0$$

$$M_{31} + M_{13} + M_{42} = 0$$

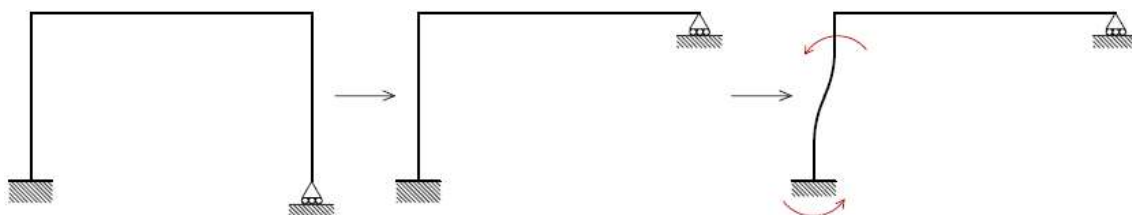
Displacement 2 E.E.



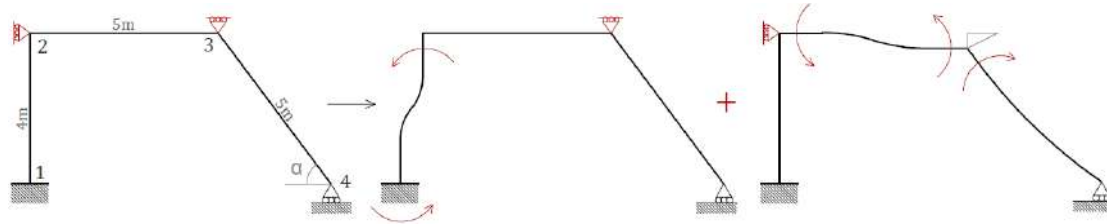
$$V_{24} = 0; \frac{M_{42}}{l_{24}} = 0$$

$$M_{42} = 0$$

When the reaction in the roller support aligns with the bar it belongs to, we can apply the following simplification. We can remove this bar from the frame. This way, we reduce one translational degree.

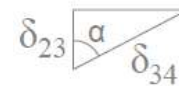


2) Trans. degree: 2

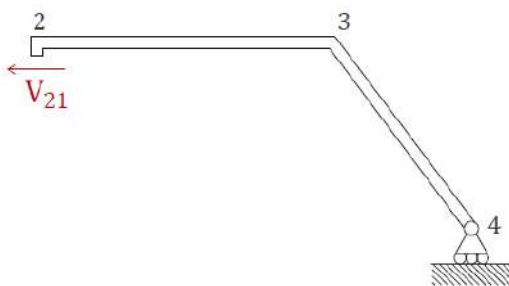


$$d = 2n - b - r = 8 - 3 - 3 = 2$$

$$n = 4 \quad r = 3 \quad b = 3$$



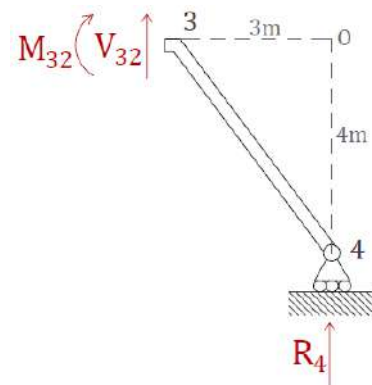
Displacement 1 E.E.



$$\Sigma H = 0; V_{21} = 0; \frac{M_{12} + M_{21}}{l_{12}} = 0;$$

$$\boxed{M_{12} + M_{21} = 0}$$

Displacement 2 E.E.



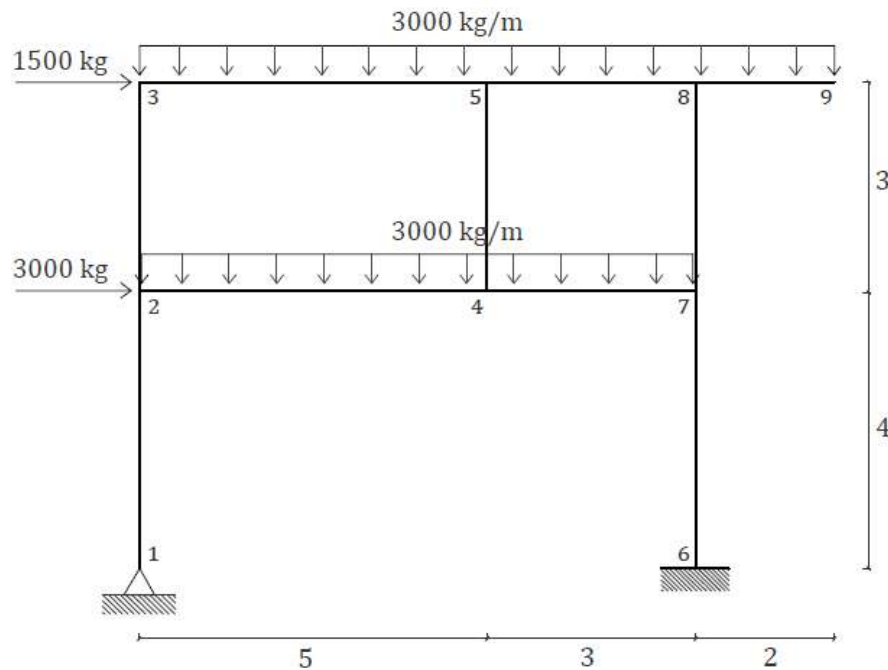
$$\Sigma M_O = 0; -3V_{32} - M_{32} = 0$$

$$3 \frac{M_{32} + M_{23}}{l_{23}} + M_{32} = 0$$

$$\boxed{1,6M_{32} + 0,6M_{23} = 0}$$

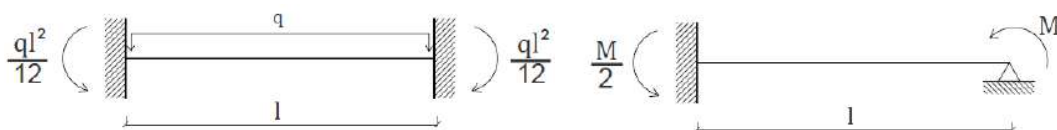
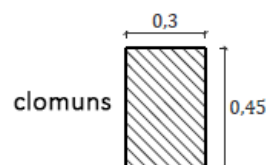
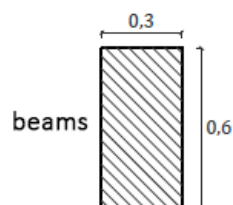
In equilibrium equations of frames with roller supports, we have to avoid involving roller support's reaction in equilibrium equations (there is not an easy way to connect it with the bending moment, as we need in this method). In the first equation of the example above we avoided it by using the horizontal forces equation (this reaction is vertical). In the second equation we have balanced moments in 0 point, collinear with this reaction.

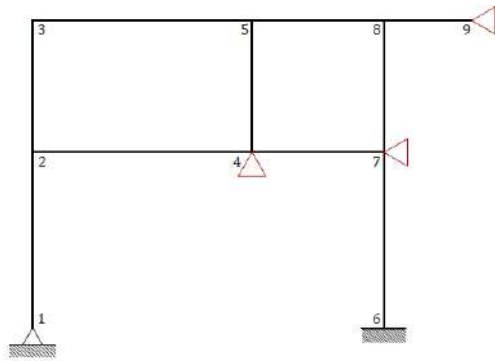
5.2.8. Sway orthogonal frame's. Example with multiple displacements



- Using Cross method, draw the bending moment, shear and axial strength diagrams.
- Solve structure's reactions.
- Solve the positive maximum bending moment of each beam.
- Calculate 3. node displacement.
- Calculate 4. node rotation.

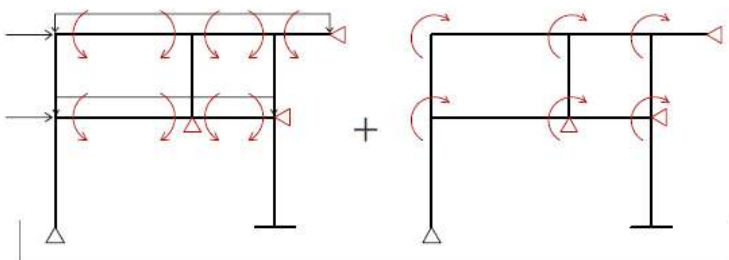
Data: $E = 200000 \text{ kg/cm}^2$; $I = (b \cdot h^3)/12$



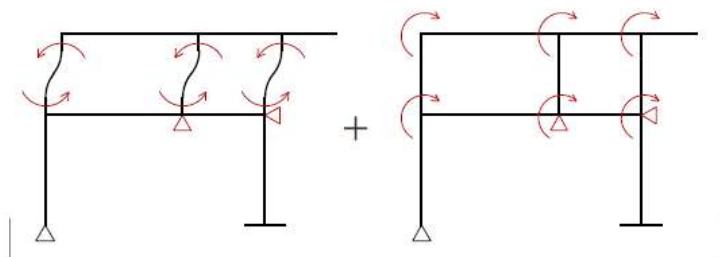


3. degree of displacement: we will need
3 virtual supports to restrain the
structure.

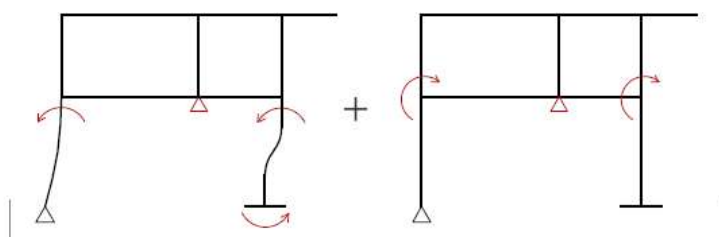
Therefore, we will need to develop 4 table.



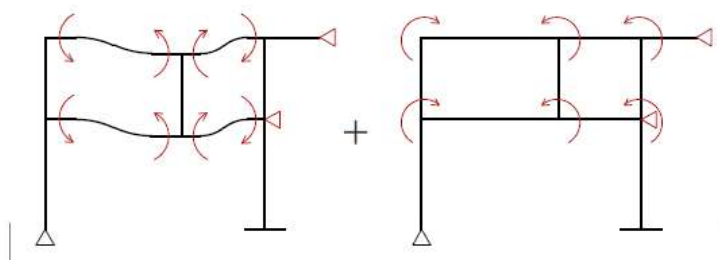
1. State
Restrained



2. State
1. displacement α_1

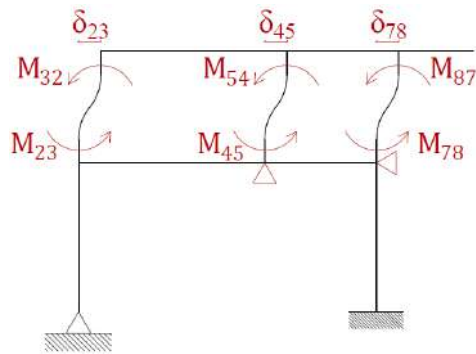


3. State
2. displacement α_2



4. State
3. displacement α_3

-1. displacement FEM



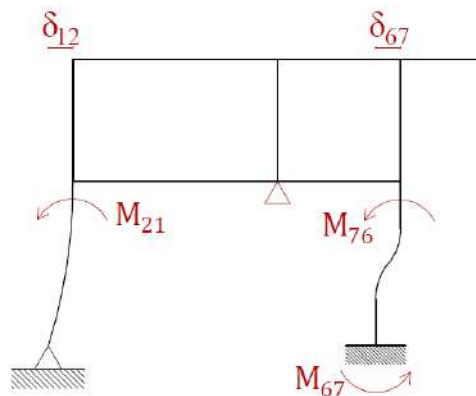
$$\delta_{23} = \delta_{45} = \delta_{78} = \Delta_1$$

$$M_{23} = M_{32} = \delta_{23} \cdot \frac{6EI}{l_{23}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{45} = M_{54} = \delta_{45} \cdot \frac{6EI}{l_{45}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

$$M_{78} = M_{87} = \delta_{78} \cdot \frac{6EI}{l_{78}^2} = 0,6 \cdot \Delta_1 \cdot EI = 1000\alpha_1$$

-2. displacement FEM



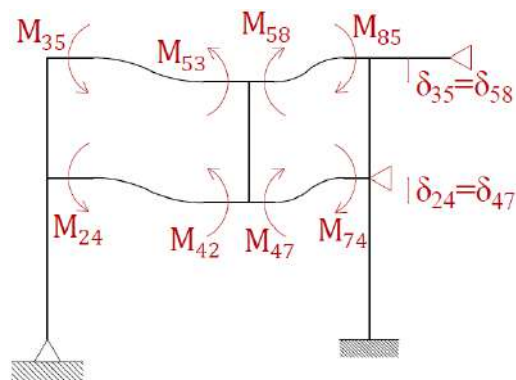
$$\delta_{12} = \delta_{78} = \Delta_2$$

$$M_{21} = \delta_{12} \cdot \frac{3EI}{l_{12}^2} = 0,1875\Delta_2 EI = 500\alpha_2$$

$$M_{67} = M_{76} = \delta_{67} \cdot \frac{6EI}{l_{67}^2} = 0,375\Delta_2 EI = 1000\alpha_2$$

$$\Delta_2 = \frac{1500\alpha_2}{EI}$$

-3. displacement FEM



$$\delta_{35} = \delta_{58} = \delta_{24} = \delta_{47} = \Delta_3$$

$$M_{24} = M_{42} = \delta_{24} \cdot \frac{6EI}{l_{24}^2} = 0,24 \cdot \Delta_3 EI = 360\alpha_3$$

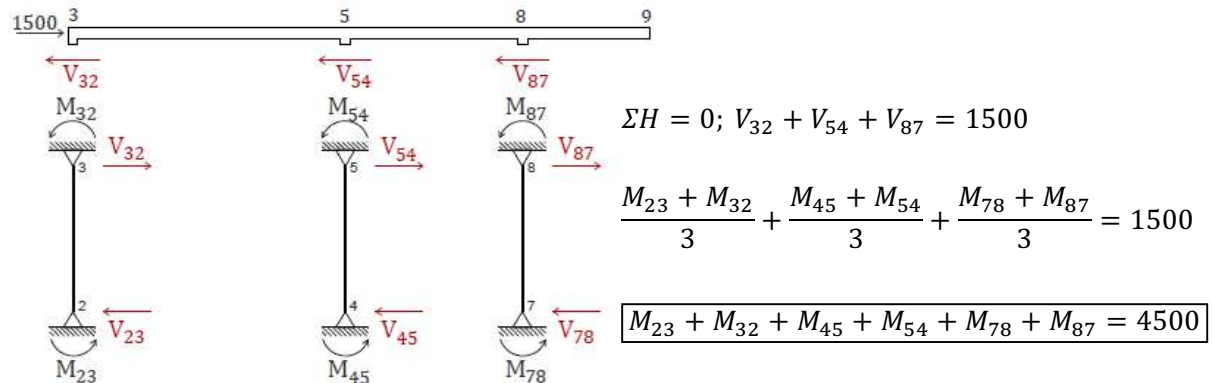
$$M_{35} = M_{53} = \delta_{35} \cdot \frac{6EI}{l_{35}^2} = 0,24 \cdot \Delta_3 \cdot EI = 360\alpha_3$$

$$M_{58} = M_{85} = -\delta_{58} \cdot \frac{6EI}{l_{58}^2} = -0,6 \cdot \Delta_3 \cdot EI = -1000\alpha_3$$

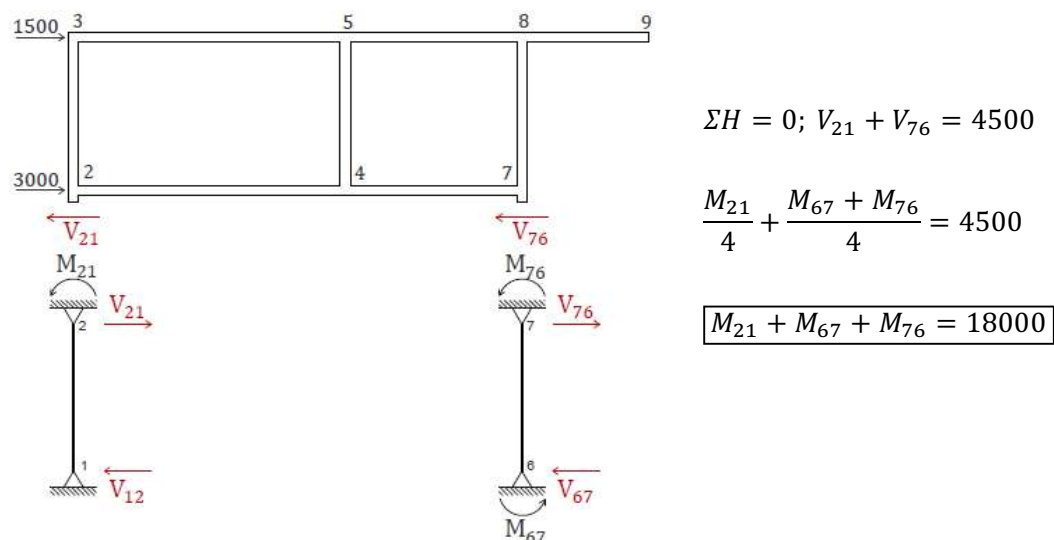
$$M_{47} = M_{74} = -\delta_{47} \cdot \frac{6EI}{l_{47}^2} = -0,6 \cdot \Delta_3 \cdot EI = -1000\alpha_3 \rightarrow \Delta_3 = \frac{1500\alpha_3}{EI}$$

Equilibrium equations

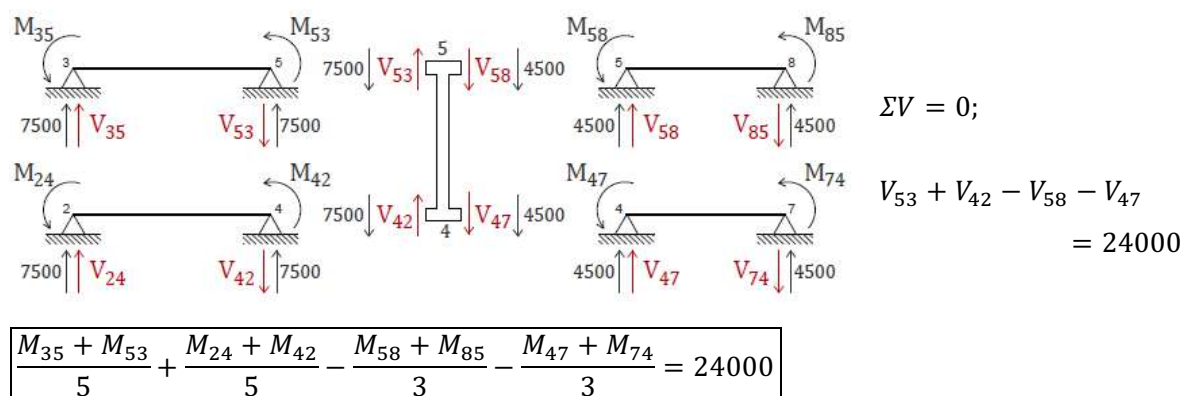
-1. displacement equilibrium equations



-2. displacement equilibrium equations



-3. displacement equilibrium equations



1. State Fixed end Moment (FEM)

$$M_{24} = -M_{42} = M_{35} = -M_{53} = \frac{ql^2}{12} = \frac{3000 \cdot 5^2}{12} = 6250 \text{ kg} \cdot \text{m}$$

$$M_{47} = -M_{74} = M_{58} = -M_{85} = \frac{ql^2}{12} = \frac{3000 \cdot 3^2}{12} = 2250 \text{ kg} \cdot \text{m}$$

$$M_{89} = \frac{ql^2}{2} = \frac{3000 \cdot 2^2}{12} = 6000 \text{ kg} \cdot \text{m}$$

Inertia

Beams: $I = \frac{bh^3}{12} = \frac{30 \cdot 60^3}{12} = 540.000 \text{ cm}^4$

Columns: $I = \frac{bh^3}{12} = \frac{30 \cdot 45^3}{12} = 227.812,5 \text{ cm}^4$

Stiffness

$$K_{21} = \frac{3EI}{l} = \frac{3 \cdot 227812,5 \cdot E}{400} = 1708,6 \cdot E$$

$$K_{76} = \frac{4EI}{l} = \frac{4 \cdot 227812,5 \cdot E}{400} = 2278,1 \cdot E$$

$$K_{23} = K_{45} = K_{78} = \frac{4EI}{l} = \frac{4 \cdot 227812,5 \cdot E}{300} = 3037,5 \cdot E$$

$$K_{24} = K_{35} = \frac{4EI}{l} = \frac{4 \cdot 540000 \cdot E}{500} = 4320 \cdot E$$

$$K_{47} = K_{58} = \frac{4EI}{l} = \frac{4 \cdot 540000 \cdot E}{300} = 7200 \cdot E$$

Distribution factors

-2. node:

$$r_{21} = \frac{K_{21}}{K_{21} + K_{23} + K_{24}} = \frac{1708,6 \cdot E}{9066,1 \cdot E} = 0,19$$

$$r_{24} = \frac{K_{24}}{\sum K_i} = \frac{4320 \cdot E}{9066,1 \cdot E} = 0,48; \quad r_{23} = 1 - r_{21} - r_{24} = 0,33$$

-3. node:

$$r_{32} = \frac{K_{23}}{K_{23} + K_{35}} = \frac{3037,5 \cdot E}{7357,5 \cdot E} = 0,41; r_{35} = 1 - r_{32} = 0,59$$

-4. node:

$$r_{42} = \frac{K_{24}}{K_{24} + K_{45} + K_{47}} = \frac{4320 \cdot E}{14557,5 \cdot E} = 0,3$$

$$r_{45} = \frac{K_{45}}{\sum K_i} = \frac{3037,5 \cdot E}{14557,5 \cdot E} = 0,21; r_{47} = 1 - r_{42} - r_{45} = 0,49$$

-5. node:

$$r_{54} = \frac{K_{45}}{K_{45} + K_{53} + K_{58}} = \frac{3037,5 \cdot E}{14557,5 \cdot E} = 0,21$$

$$r_{53} = \frac{K_{53}}{\sum K_i} = \frac{4320 \cdot E}{14557,5 \cdot E} = 0,3; r_{58} = 1 - r_{54} - r_{53} = 0,49$$

-7. node:

$$r_{76} = \frac{K_{76}}{K_{76} + K_{74} + K_{78}} = \frac{2278,1 \cdot E}{12515,6 \cdot E} = 0,18$$

$$r_{74} = \frac{K_{47}}{\sum K_i} = \frac{7320 \cdot E}{12515,6 \cdot E} = 0,58; r_{78} = 1 - r_{76} - r_{74} = 0,24$$

-8. node:

$$r_{87} = \frac{K_{87}}{K_{87} + K_{85}} = \frac{3037,5 \cdot E}{10237,5 \cdot E} = 0,3; r_{85} = 1 - r_{87} = 0,7$$

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3. STATE (DISP. 2)																	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
12	-	0.19	0.34	0.48	0.30	0.21	0.49	0.58	0.24	0.18	1000.0	0.41	0.59	0.30	0.21	0.49	0.70
		500.0	500.0														
		-500.0	-94.2	-167.5	-238.3	-119.1	-119.1	1029.5	29.5			-83.8			12.4		-124.9
					17.7	119.1	35.4	24.9	59.9	1029.5	29.5						
									-206.1	-1029.5	-592.2	-249.8	-187.4	-93.7	-83.8		
		35.0		17.3											83.8	34.6	49.2
		-35.0	-6.6	-11.7	-16.7	-308.3	-8.3	-3.9	152.5	96.1	76.2	19.9			-5.9	-5.5	-37.0
					45.7	308.3	91.5	64.3	-27.7	-96.1	-55.3	-23.3	-17.5	-8.7			
		48.1		2.3											-11.4	4.7	6.7
		-48.1	-9.1	-16.1											11.4	-8.1	-12.3

Equation resolution

$$1) M_{23} + M_{32} + M_{45} + M_{54} + M_{78} + M_{87} = 4500$$

$$M_{23} = -3167,8 + 582\alpha_1 - 175,3\alpha_2 - 193,6\alpha_3$$

$$M_{32} = -3582,1 + 539,3\alpha_1 - 49,8\alpha_2 - 209\alpha_3$$

$$M_{45} = 1599,9 + 838,2\alpha_1 + 86,4\alpha_2 + 139,6\alpha_3$$

$$M_{54} = 1982,7 + 853,6\alpha_1 + 19,9\alpha_2 + 134,8\alpha_3$$

$$M_{78} = -347,3 + 700,7\alpha_1 - 252,3\alpha_2 + 305,3\alpha_3$$

$$M_{87} = -1482,4 + 670,9\alpha_1 - 87\alpha_2 + 331,6\alpha_3$$

$$-4997 + 4184,7\alpha_1 - 458,1\alpha_2 + 508,7\alpha_3 = 4500$$

$$-9497 + 4184,7\alpha_1 - 458,1\alpha_2 + 508,7\alpha_3 = 0$$

$$2) M_{21} + M_{67} + M_{76} = 18000$$

$$M_{21} = -1045,5 - 141,7\alpha_1 + 388\alpha_2 - 66,1\alpha_3$$

$$M_{67} = 389,8 - 135\alpha_1 + 791,5\alpha_2 + 139,8\alpha_3$$

$$M_{76} = 194,9 - 67,5\alpha_1 + 895,7\alpha_2 + 69,9\alpha_3$$

$$-460,8 - 344,2\alpha_1 + 2075,2\alpha_2 + 143,6\alpha_3 = 18000$$

$$-18460,8 - 344,2\alpha_1 + 2075,2\alpha_2 + 143,6\alpha_3 = 0$$

$$3) \frac{M_{35} + M_{53}}{5} + \frac{M_{24} + M_{42}}{5} - \frac{M_{58} + M_{85}}{3} - \frac{M_{47} + M_{74}}{3} = 24000$$

$$M_{35} = 3582,1 - 539,3\alpha_1 + 49,8\alpha_2 + 209\alpha_3$$

$$M_{53} = -5901,8 - 362,8\alpha_1 - 8,2\alpha_2 + 376,9\alpha_3$$

$$M_{24} = 4213,3 - 440,2\alpha_1 - 212,7\alpha_2 + 259,7\alpha_3$$

$$M_{42} = -6401,2 - 345,9\alpha_1 + 2,3\alpha_2 + 412,3\alpha_3$$

$$M_{58} = 3919 - 490,8\alpha_1 - 11,7\alpha_2 - 511,7\alpha_3$$

$$M_{85} = -4517,6 - 670,9\alpha_1 - 87\alpha_2 - 331,6\alpha_3$$

$$M_{47} = 4801,3 - 492,3\alpha_1 - 88,7\alpha_2 - 551,9\alpha_3$$

$$M_{74} = -42,5 - 565,7\alpha_1 - 539,2\alpha_2 - 445,1\alpha_3$$

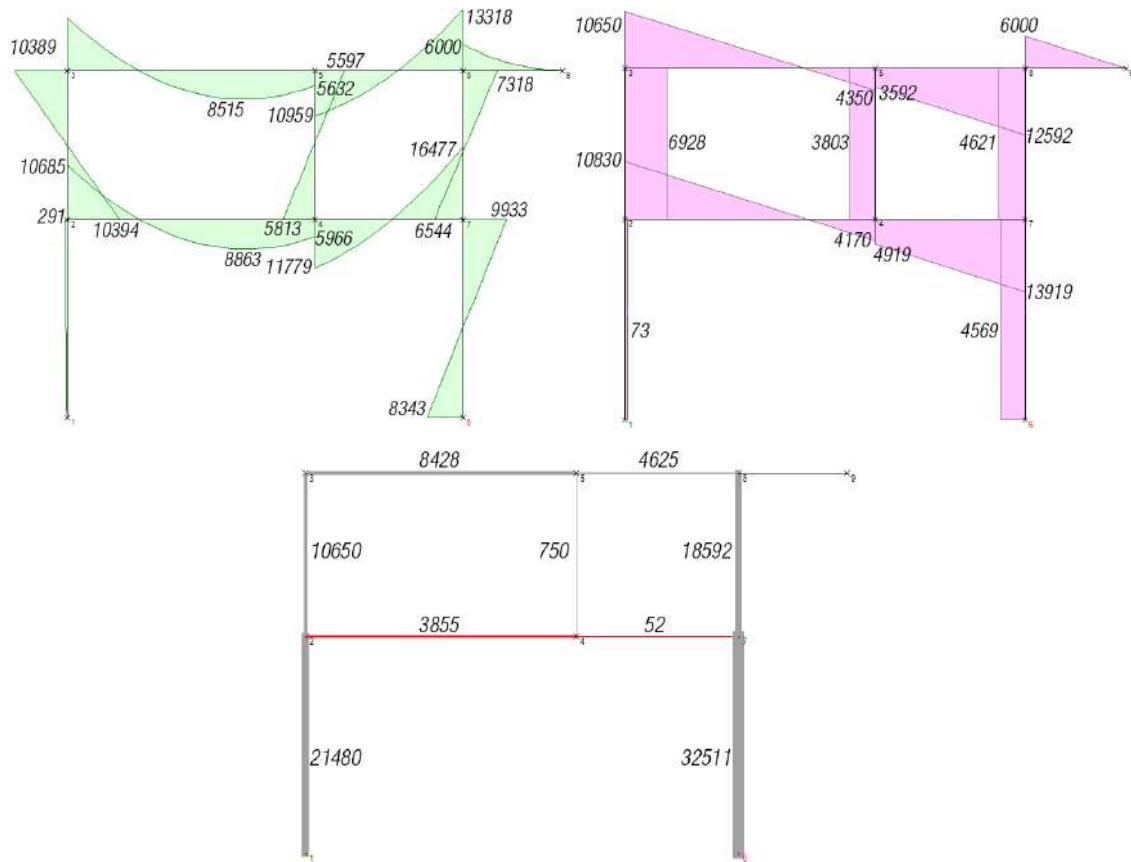
$$-2288,25 + 402,26\alpha_1 - 150,44\alpha_2 + 865\alpha_3 = 24000$$

$$-26288,25 + 402,26\alpha_1 - 150,44\alpha_2 + 865\alpha_3 = 0$$

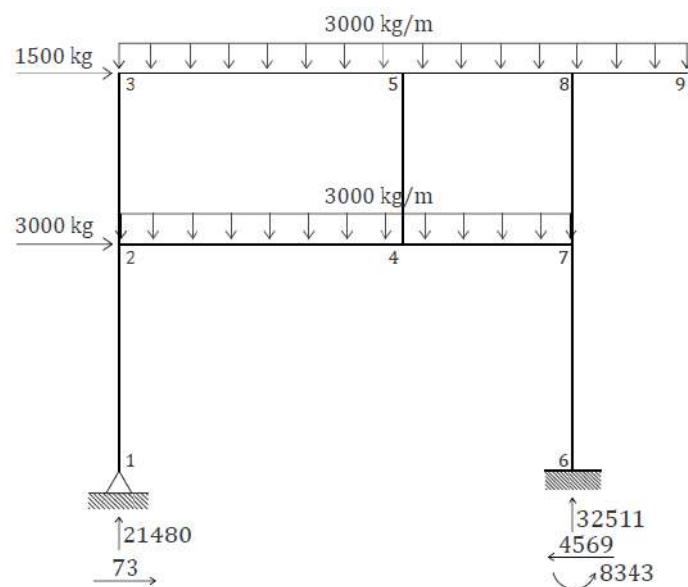
Equation system resolution

$$\left. \begin{array}{l} -9497 + 4184,7\alpha_1 - 458,1\alpha_2 + 508,7\alpha_3 = 0 \\ -18460,8 - 344,2\alpha_1 + 2075,2\alpha_2 + 143,6\alpha_3 = 0 \\ -26288,25 + 402,26\alpha_1 - 150,44\alpha_2 + 865\alpha_3 = 0 \end{array} \right\} \begin{array}{l} \alpha_1 = -0,58 \\ \alpha_2 = 6,75 \\ \alpha_3 = 29,47 \end{array}$$

Diagrams



Reactions



$$\Sigma H = 0; 3000 + 1500 = 4569 - 73; 4500 \cong 4496 \checkmark$$

$$\Sigma V = 0; 3000 \cdot 10 + 3000 \cdot 8 = 21480 + 32511; 54000 \cong 53991 \checkmark$$

$$\Sigma M_1 = 0; 3000 \cdot 10 \cdot 5 + 3000 \cdot 8 \cdot 4 + 1500 \cdot 7 + 3000 \cdot 4 = 32511 \cdot 8 + 8343$$

$$150000 + 96000 + 10500 + 12000 = 260088 + 8343; 268500 \cong 268431 \checkmark$$

3. node displacement ($E = 200000 \text{ kg/cm}^2$)

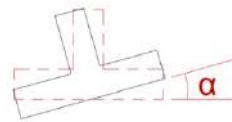
$$\begin{aligned} \delta_3 &= \frac{M_{32} \cdot l_{32}^2}{6EI} + \frac{M_{21} \cdot l_{21}^2}{3EI} = \frac{1000\alpha_1 \cdot l_{32}^2}{6 \cdot 2 \cdot 10^5 \cdot 2,28 \cdot 10^5} + \frac{500\alpha_2 \cdot l_{21}^2}{3 \cdot 2 \cdot 10^5 \cdot 2,28 \cdot 10^5} = \\ &= \frac{-58000 \cdot 300^2}{2,736 \cdot 10^{11}} + \frac{337500 \cdot 400^2}{1,368 \cdot 10^{11}} = 0,37 \text{ cm} \end{aligned}$$

4. node rotation (calculated in 4-2 end)

$$\alpha = \frac{M \cdot l}{4EI} = \frac{624103 \cdot 500}{4 \cdot 2 \cdot 10^5 \cdot 5,4 \cdot 10^5} = 7,2 \cdot 10^{-4} \text{ rad}$$

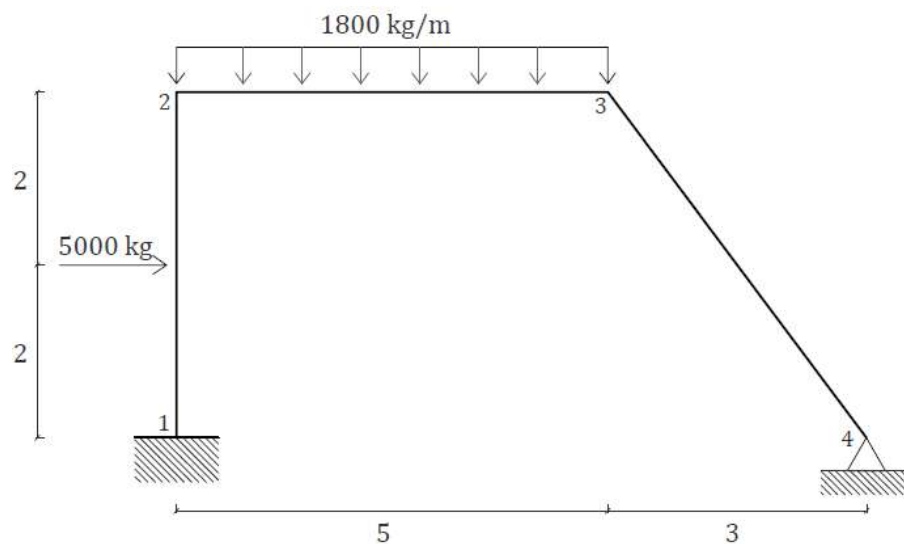
$$\begin{aligned} M &= 1171 - 1676\alpha_1 + 143\alpha_2 + 136\alpha_3 = 1171 + 96,86 + 965,25 + 4007,92 = \\ &6241,03 \text{ kg} \cdot \text{m} = 624103 \text{ kg} \cdot \text{cm} \end{aligned}$$

$$360^\circ \rightarrow 2\pi; x \rightarrow 7,2 \cdot 10^{-4}; \alpha = 0,04^\circ$$

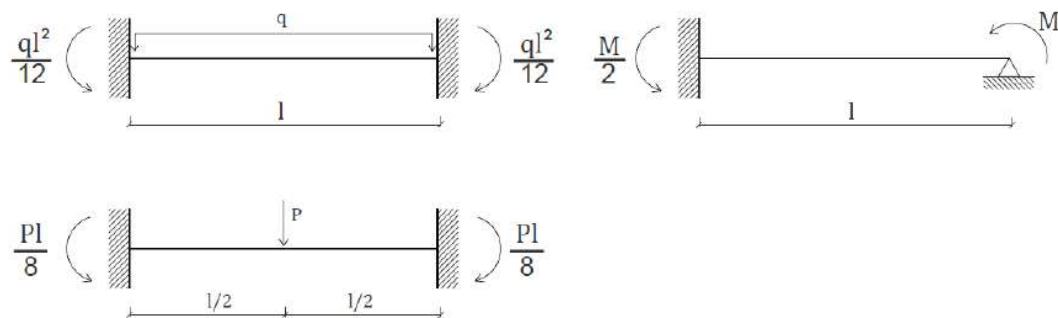


5.2.9. Sway frames with inclined bars. Examples

Using Cross method, solve the structure's reactions and draw the bending moment, shear and axial strength diagrams.



Data: $EI = \text{constant}$



Fixed end Moment (FEM)

$$M_{12} = M_{21} = \frac{Pl}{8} = \frac{5000 \cdot 4}{8} = 2500 \text{ kg} \cdot \text{m}$$

$$M_{23} = M_{34} = \frac{ql^2}{12} = \frac{1800 \cdot 5^2}{12} = 3750 \text{ kg} \cdot \text{m}$$

Stiffness

$$K_{21} = \frac{4EI}{l} = \frac{4EI}{4} = EI$$

$$K_{23} = \frac{4EI}{l} = \frac{4EI}{5} = 0,8 EI$$

$$K_{34} = \frac{3EI}{l} = \frac{3EI}{5} = 0,6 EI$$

Distribution factors

-2. node:

$$r_{21} = \frac{K_{21}}{K_{21} + K_{23}} = \frac{EI}{(1 + 0,8)EI} = 0,5$$

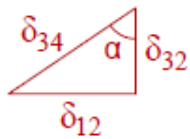
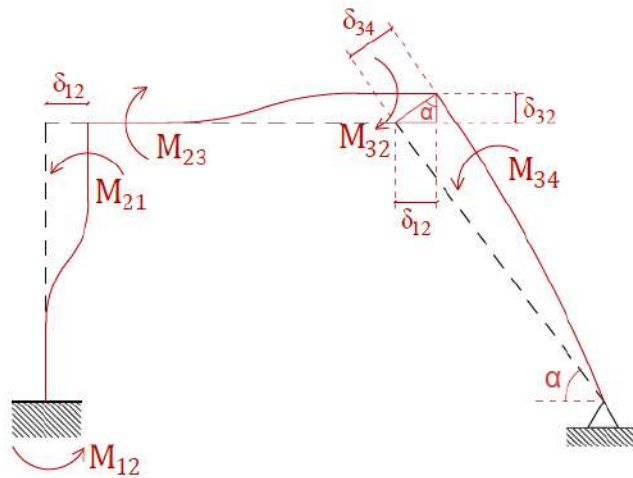
$$r_{23} = \frac{K_{23}}{K_{21} + K_{23}} = \frac{0,8 EI}{(1 + 0,8)EI} = 0,4$$

-3. node:

$$r_{32} = \frac{K_{23}}{K_{23} + K_{34}} = \frac{0,8 EI}{(0,8 + 0,6)EI} = 0,57$$

$$r_{34} = \frac{K_{34}}{K_{23} + K_{34}} = \frac{0,6 EI}{(0,8 + 0,6)EI} = 0,43$$

2. Cross FEM



$$\sin \alpha = \frac{4}{5} = 0,8 ; \sin \alpha = \frac{\delta_{12}}{\delta_{34}} ; \delta_{12} = 0,8 \cdot \delta_{34}$$

$$\cos \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{\delta_{32}}{\delta_{34}} ; \delta_{32} = 0,6 \cdot \delta_{34}$$

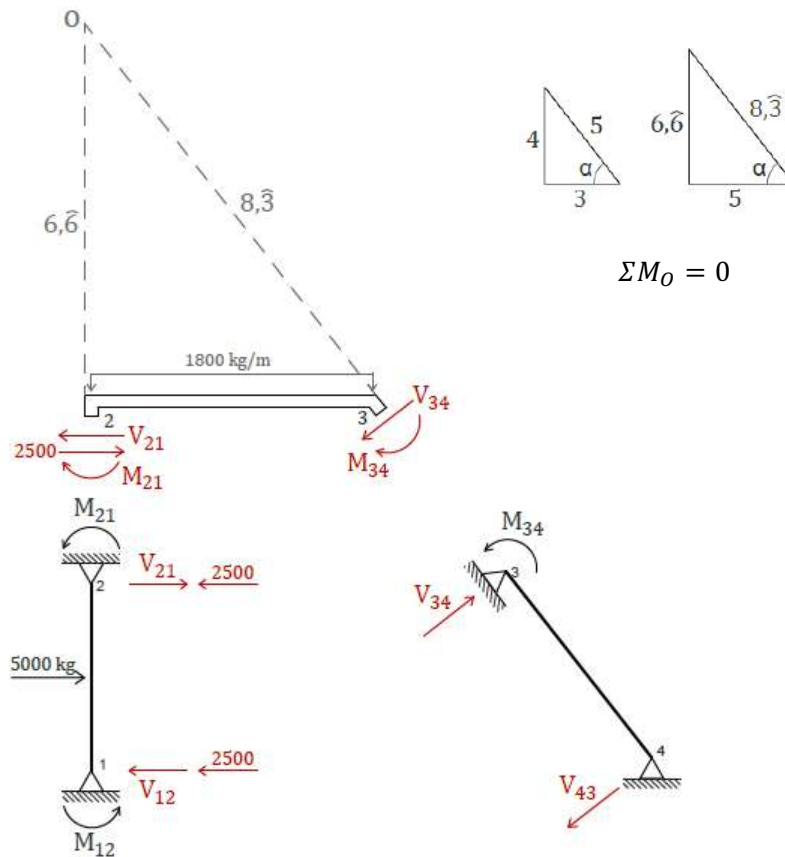
$$\delta_{34} = \Delta ; \delta_{12} = 0,8 \Delta ; \delta_{32} = 0,6 \Delta$$

$$M_{12} = M_{21} = \delta_{12} \cdot \frac{6EI}{l_{12}^2} = 0,8 \cdot \Delta \cdot \frac{6EI}{4^2} = 0,3 \cdot \Delta \cdot EI = 1000\alpha \rightarrow \Delta = \frac{3333,3\alpha}{EI}$$

$$M_{23} = M_{32} = -\delta_{32} \cdot \frac{6EI}{l_{32}^2} = -0,6 \cdot \Delta \cdot \frac{6EI}{5^2} = -0,144 \cdot \Delta \cdot EI = -0,144 \cdot 3333,3\alpha \cdot \frac{EI}{EI} = -480\alpha$$

$$M_{34} = \delta_{34} \cdot \frac{3EI}{l_{34}^2} = \Delta \cdot \frac{3EI}{5^2} = 0,12 \cdot \Delta \cdot EI = 0,12 \cdot 3333,3\alpha \cdot \frac{EI}{EI} = 400\alpha$$

Displacement equilibrium equation



$$\Sigma M_O = 0$$

$$1800 \cdot 5 \cdot 2,5 - 2500 \cdot 6,6 + V_{21} \cdot 6,6 + V_{34} \cdot 8,3 + M_{21} + M_{34} = 0$$

$$5833,3 + 6,6 \frac{M_{21} + M_{12}}{4} + 8,3 \frac{M_{34}}{5} + M_{21} + M_{34} = 0$$

$$5833,3 + 2,6M_{21} + 1,6M_{12} + 2,6M_{34} = 0$$

Using the moments of Cross table:

$$M_{12} = 1811,5 + 839\alpha$$

$$M_{21} = -3876,9 + 678\alpha$$

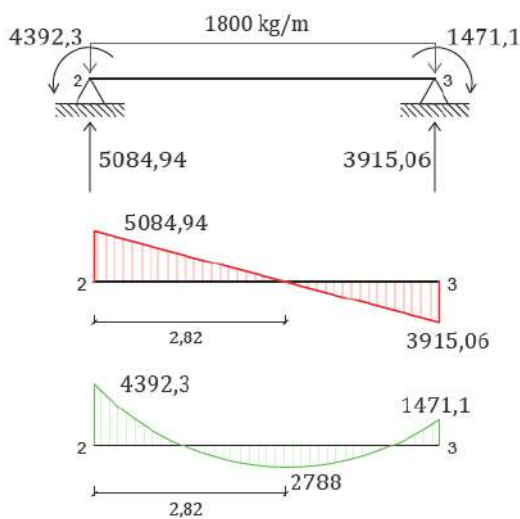
$$M_{34} = 1843,2 + 489,5\alpha$$

$$5833,3 + 2,6(-3876,9 + 678\alpha) + 1,6(1811,5 + 839\alpha) + 2,6(1843,2 + 489,5\alpha) = 0$$

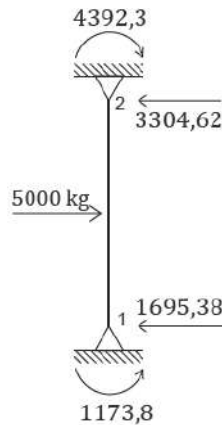
$$3429,20 + 4511,55\alpha = 0 ; \alpha = -\frac{3429,20}{4511,55} = -0,76$$

Cross table

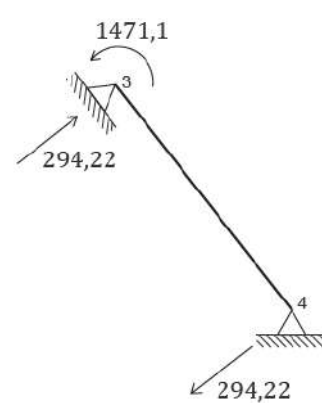
1.STATE (RESTRAINED)	1	2		3		4	
	12	21	23	32	34	43	
	-	0,556	0,444	0,571	0,429	-	
FEM	2500,00	1250,00	-2500,00	3750,00	-3750,00		
	-347,22	-1250,00	-694,44	-555,56	-4027,78	-277,78	
		1150,79	1150,79	4027,78	2301,59	1726,19	
	-319,66	-1150,79	-639,33	-511,46	-255,73	-255,73	
		73,07	73,07	255,73	146,13	109,60	
	-20,30	-73,07	-40,59	-32,47	-16,24	-16,24	
		4,64	4,64	16,24	9,28	6,96	
	-1,29	-4,64	-2,58	-2,06	-1,03	-1,03	
				1,03	0,59	0,44	
	1811,5	-3876,9	3876,9	-1843,2	1843,2	0,0	
2.STATE (SWAY FRAME)	1	2		3		4	
	12	21	23	32	34	43	
	-	0,56	0,44	0,57	0,43	-	
FEM x α	1000,00	520,00	1000,00	-480,00	-480,00	400,00	
	-144,44	-520,00	-288,89	-231,11	-195,56	-115,56	
		55,87	55,87	195,56	111,75	83,81	
	-15,52	-55,87	-31,04	-24,83	-12,42	-12,42	
		3,55	3,55	12,42	7,09	5,32	
	-0,99	-3,55	-1,97	-1,58	-0,79	-0,79	
		0,23	0,23	0,79	0,45	0,34	
	-0,06	-0,23	-0,13	-0,10	-0,05	-0,05	
				0,05	0,03	0,02	
	839,0	678,0	-678,0	-489,5	489,5	0,0	
1	1	2		3		4	
	12	21	23	32	34	43	
1. STATE	1811,5	-3876,9	3876,9	-1843,2	1843,2	0,0	
2. STATE x α; α= -0,76	-637,7	-515,3	515,3	372,1	-372,1	0,0	
FINAL RESULTS	1173,8	-4392,3	4392,3	-1471,1	1471,1	0,0	



2-3 bar

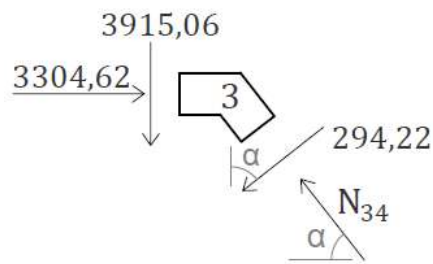


1-2 bar



3-4 bar

$$x = \frac{5084,94}{1800} = 2,82; M_{Max} = \Delta_{0-2,82} - M_{23} = 2788 \text{ kg} \cdot \text{m}$$



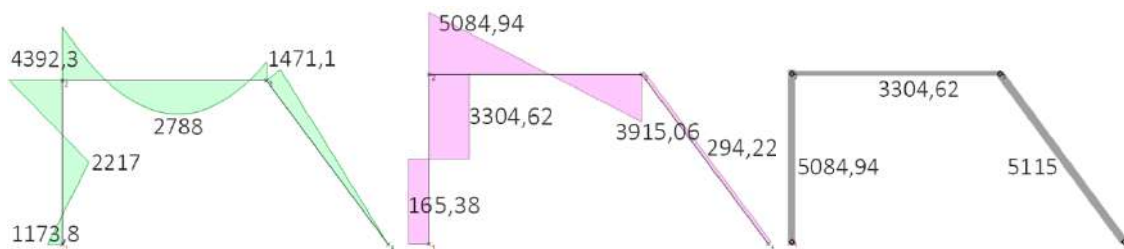
$$\Sigma H = 0 ; 3304,62 = 294,22 \sin \alpha + N_{34} \cdot \cos \alpha$$

$$N_{34} = \frac{3304,62 - 294,22 \cdot 0,8}{0,6} ; N_{34} \approx 5115 \text{ kg}$$

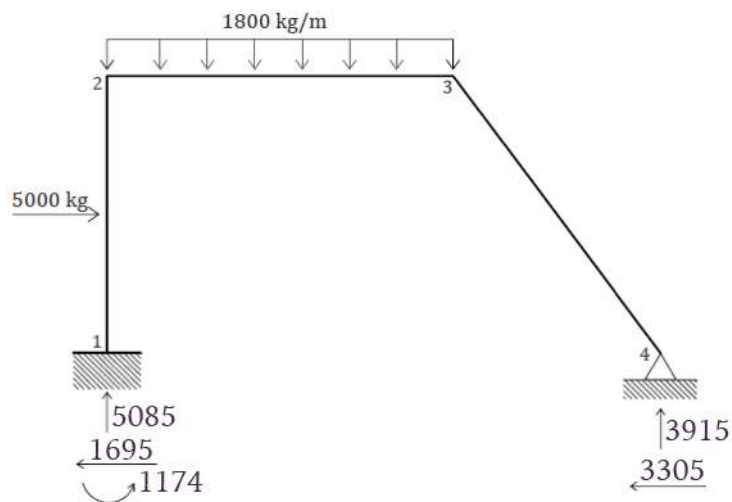
$$\Sigma V = 0 ; 3915,06 + 294,22 \cos \alpha = N_{34} \cdot \sin \alpha$$

$$N_{34} = \frac{4091,59}{0,8} \approx 5115 \text{ kg}$$

Diagrams



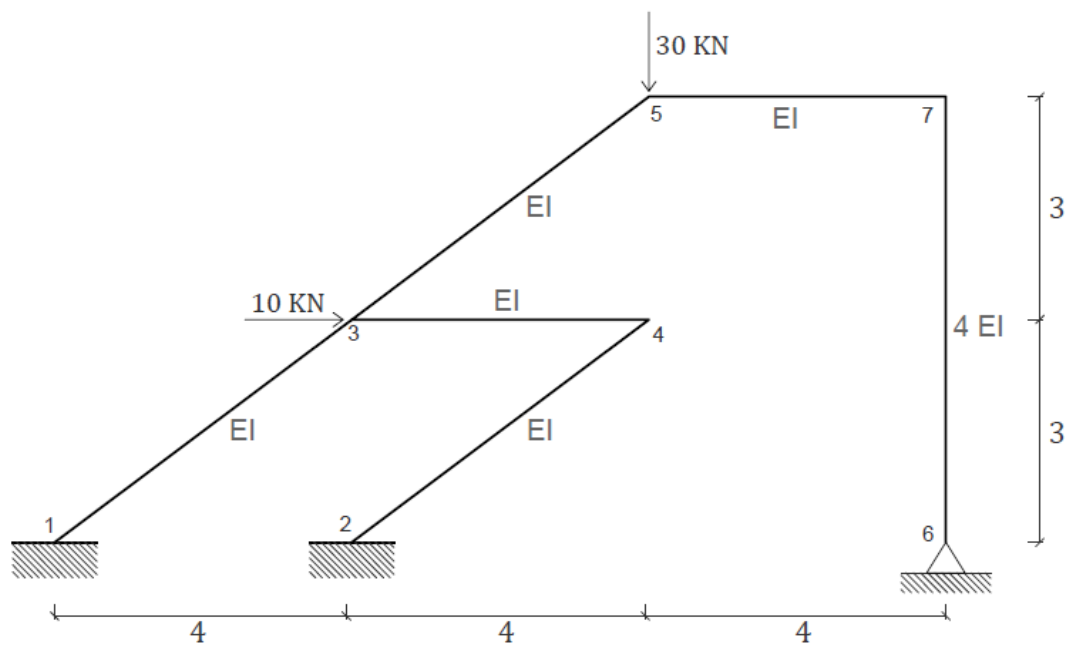
Reactions



$$\Sigma H = 0 ; 5000 = 1695 + 3305 \checkmark$$

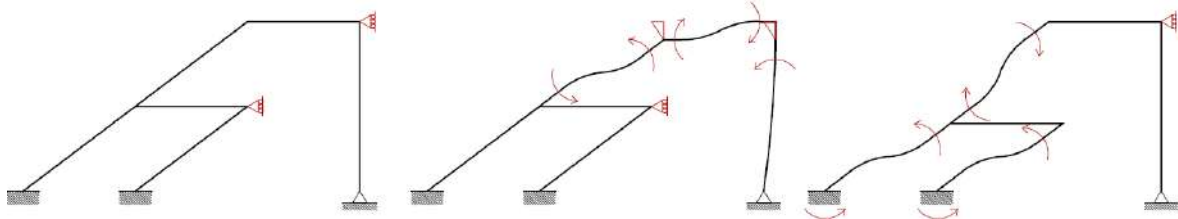
$$\Sigma V = 0 ; 1800 \cdot 5 = 5085 + 3915 ; 9000 = 9000 \checkmark$$

$$\Sigma M_1 = 0 ; 5000 \cdot 2 + 1800 \cdot 5 \cdot 2,5 = 3915 \cdot 8 + 1184 ; 32500 \cong 32504 \checkmark$$



- Using Cross method, draw the bending moment, shear and axial strength diagrams.
- Draw the deformation.

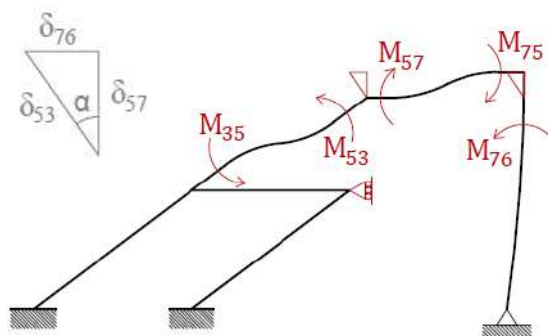
Translational degree



$$d = 2n - b - r = 2 \cdot 7 - 6 - 6 = 2$$

$$n = 7 ; b = 6 ; r = 6$$

1. Displacement FEM



$$\sin \alpha = 0,6 ; \cos \alpha = 0,8$$

$$\delta_{53} = \Delta_1 ; \delta_{57} = 0,8 \cdot \Delta_1 ; \delta_{76} = 0,6 \cdot \Delta_1$$

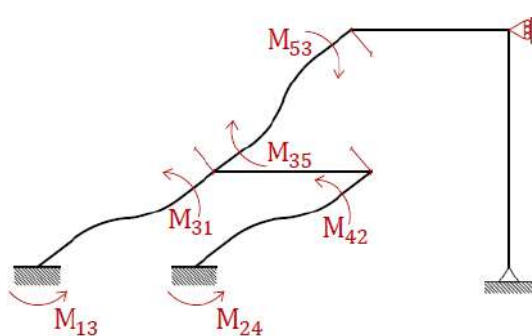
$$M_{35} = M_{53} = \delta_{53} \frac{6EI}{l_{35}^2} = 0,24 \Delta_1 EI = 800\alpha_1$$

$$M_{76} = \delta_{76} \frac{3 \cdot (4EI)}{l_{76}^2} = 0,2 \Delta_1 EI = 666,6\alpha_1$$

$$M_{57} = M_{75} = -\delta_{57} \frac{6EI}{l_{57}^2} = -0,3 \Delta_1 EI = -1000\alpha_1 \rightarrow \Delta_1 = \frac{3333,3\alpha_1}{EI}$$

$$M_{35} = M_{53} = 800\alpha_1 ; M_{76} = 666,6\alpha_1 ; M_{57} = M_{75} = -1000\alpha_1$$

2. Displacement FEM



$$\delta_{31} = \delta_{42} = \delta_{35} = \Delta_2$$

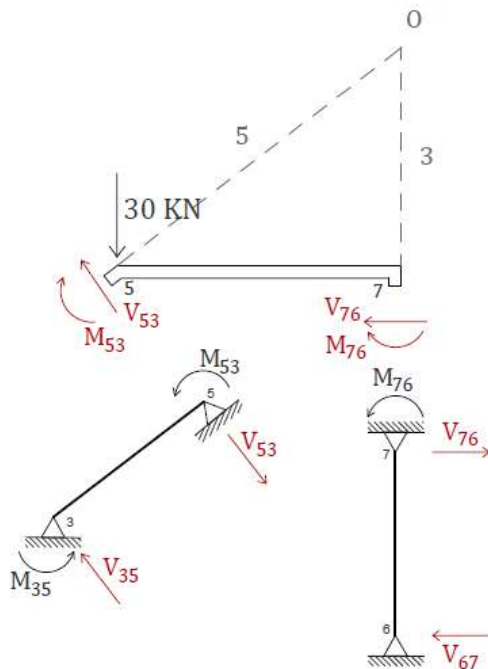
$$M_{31} = M_{13} = \delta_{31} \frac{6EI}{l_{31}^2} = 0,24 \Delta_2 EI = 1000\alpha_2$$

$$M_{35} = M_{53} = -\delta_{35} \frac{6EI}{l_{35}^2} = -0,24 \Delta_2 EI = -1000\alpha_2$$

$$M_{24} = M_{42} = \delta_{42} \frac{6EI}{l_{42}^2} = 0,24 \Delta_2 EI = 1000\alpha_2$$

$$M_{31} = M_{13} = 1000\alpha_2 ; M_{35} = M_{53} = -1000\alpha_2 ; M_{24} = M_{42} = 1000\alpha_2$$

1. displacement balance equation



$$\Sigma M_0 = 0$$

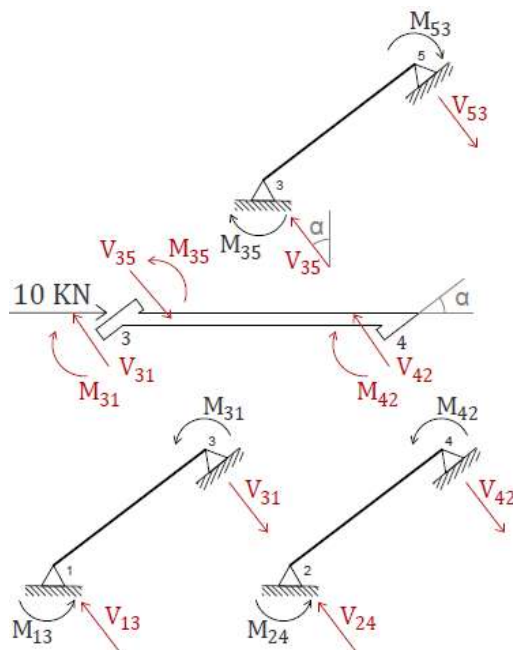
$$30 \cdot 4 - M_{53} - M_{76} - 5 \cdot V_{53} - 3 \cdot V_{76} = 0$$

$$120 - M_{53} - M_{76} - 5 \frac{M_{53} + M_{35}}{5} - 3 \frac{M_{76}}{6} = 0$$

$$120 - 2 M_{53} - M_{35} - 1,5 M_{76} = 0$$

$$\rightarrow 3419,44\alpha_1 - 2102,77\alpha_2 = 120$$

2. displacement balance equation



$$\Sigma V = 0$$

$$V_{35} + 10 \cdot \sin \alpha - V_{31} - V_{42} = 0$$

$$\frac{M_{35} + M_{53}}{5} + 10 \cdot 0,6 - \frac{M_{31} + M_{13}}{5} - \frac{M_{24} + M_{42}}{5} = 0$$

$$M_{35} + M_{53} + M_{31} + M_{13} + M_{24} + M_{42} = 30$$

$$\rightarrow -1682,28\alpha_1 + 4633,93\alpha_2 = 30$$

Stiffness

$$K_{13} = K_{24} = K_{35} = \frac{4 EI}{l} = \frac{4 EI}{5} = 0,8 EI ; K_{34} = K_{57} = \frac{4 EI}{l} = \frac{4 EI}{4} = EI$$

$$K_{76} = \frac{3 (4 EI)}{l} = \frac{3 (4 EI)}{6} = 2 EI$$

Distribution factors

3. node

$$r_{31} = r_{35} = \frac{K_{13}}{K_{13} + K_{35} + K_{34}} = \frac{0,8 EI}{2,6 EI} = 0,308 ; r_{34} = 1 - r_{31} - r_{35} = 0,384$$

4. node

$$r_{43} = \frac{K_{34}}{K_{34} + K_{24}} = \frac{EI}{1,8 EI} = 0,5 ; r_{42} = 1 - r_{43} = 0,5$$

5. node

$$r_{53} = \frac{K_{35}}{K_{35} + K_{57}} = \frac{0,8 EI}{1,8 EI} = 0,4 ; r_{57} = 1 - r_{53} = 0,6$$

7. node

$$r_{75} = \frac{K_{57}}{K_{57} + K_{76}} = \frac{EI}{3 EI} = 0,3 ; r_{76} = 1 - r_{75} = 0,7$$

Equation resolution

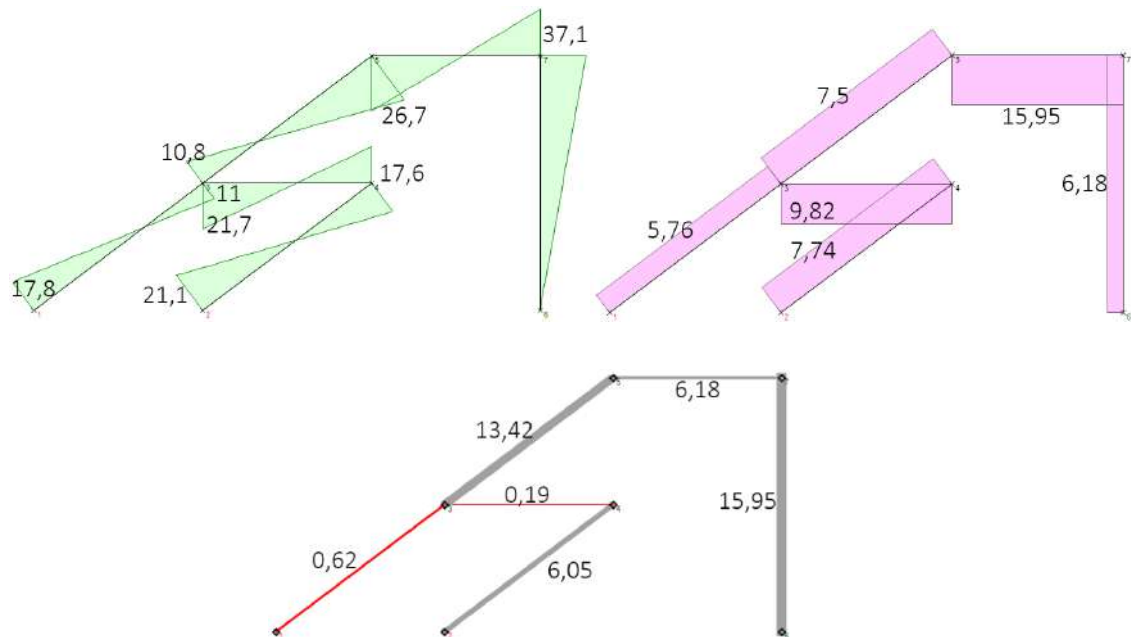
$$1) 120 - 2 M_{53} - M_{35} - 1,5 M_{76} = 0 ; \boxed{3419,44\alpha_1 - 2102,77\alpha_2 = 120}$$

$$2) M_{35} + M_{53} + M_{31} + M_{13} + M_{24} + M_{42} = 30 ; \boxed{-1682,28\alpha_1 + 4633,93\alpha_2 = 30}$$

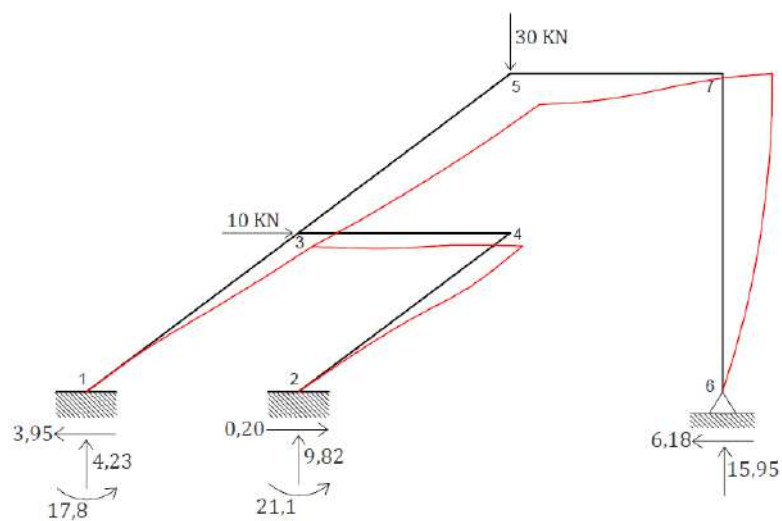
$$\boxed{\alpha_1 = 0,05 ; \alpha_2 = 0,0247}$$

1.STATE (1.DISP)		3		4		2		5		7		6	
1		31	34	35	43	42	24	53	57	75	76	6	67
13													
-		0,308	0,385	0,308	0,556	0,444	-	0,444	0,556	0,333	0,667	-	-
EFM x α_1		800,0		800,0				800,0	-1000,0	-1000,0	666,7		
-123,1		-246,2	-307,7	-246,2	-153,8			-323,1	-123,1				
		114,5	42,7	71,8	153,8	68,4	34,2	323,1	143,6				
-17,6		-35,2	-44,0	-35,2	-17,6			40,6	40,6				
		-4,1	4,9	-9,0	17,6	7,8	3,9	-40,6	-18,0				
0,6		1,3	1,6	1,3	0,6			1,9	1,9				
					-0,6	-0,3	-0,1	-1,9	-0,8				
									-1,0				
									0,5	0,2	0,3		
-140,06		-280,12	-302,53	582,65	-75,92	75,92	37,96	801,63	-801,63	-836,93	836,93	0,00	0,00
2.STATE (2.DISP)		3		4		2		5		7		6	
1		31	34	35	43	42	24	53	57	75	76	6	67
13													
-		0,308	0,385	0,308	0,556	0,444	-	0,444	0,556	0,333	0,667	-	-
EFM x α_2		1000,0		-1000,0		1000,0	1000,0	-1000,0					
0,0		0,0	0,0	0,0	1000,0			-1000,0	0,0				
		-55,6	-277,8	222,2	-1000,0	-444,4	-222,2	1000,0	444,4				
8,5		55,6	21,4	17,1	8,5			-46,3	-46,3				
		7,9	-2,4	10,3	-8,5	-3,8	-1,9	46,3	20,6				
-1,2		-7,9	-3,0	-2,4	-1,2			-2,1	2,1				
					1,2	0,5	0,3	2,1	1,0				
1007,33		1014,66	-261,83	-752,83	-552,30	552,30	776,15	-534,03	534,03	194,16	-194,16	0,00	0,00
FINAL RESULTS		3		4		2		5		7		6	
1		31	34	35	43	42	24	53	57	75	76	6	67
13													
-7,01	$\alpha_1 = 0,050$	-14,02	-15,14	29,15	-3,80	3,80	1,90	40,11	-40,11				
24,89	$\alpha_2 = 0,025$	25,07	-6,47	-18,60	-13,65	13,65	19,18	-13,20	13,20				
17,88		11,06	-21,61	10,55	-17,45	17,45	21,08	26,91	-26,91	-37,08	37,08	0,00	0,00

Diagrams



Reactions and deformation

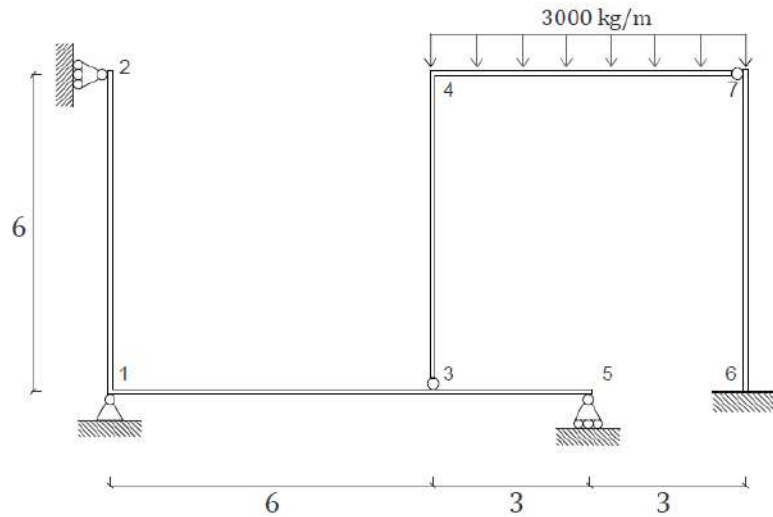


$$\sum H = 0 ; 10 = 3,95 + 6,18 - 0,2 ; 10 \cong 10 \checkmark$$

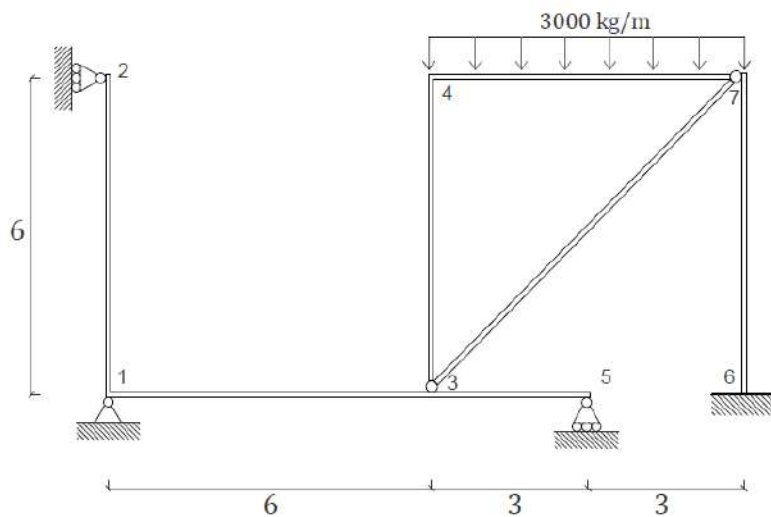
$$\sum V = 0 ; 30 = 4,23 + 9,82 + 15,95 ; 30 = 30 \checkmark$$

$$\sum M_1 = 0 ; -10 \cdot 3 - 30 \cdot 8 + 17,8 + 21,1 + 9,82 \cdot 4 + 15,95 \cdot 12 = 0 ; 269,58 \cong 270 \checkmark$$

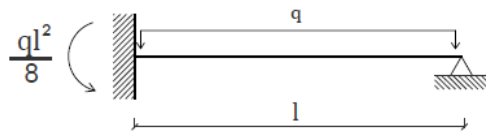
5.2.10. Sway frames with hinged nodes. Example



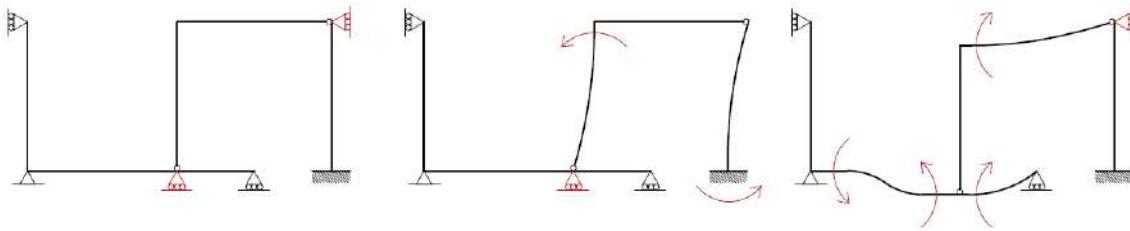
- Using Cross method, draw the bending moment, shear and axial strength diagrams. Calculate the reactions and draw the deformation.
- Calculate the horizontal displacement of 7 node and vertical displacement of 3 node
- If we add a diagonal bar to the frame: calculate the translational degree, displacement's FEM and equilibrium equations.



Data: $EI = \text{constant} = 5,284 \cdot 10^6 \text{ kg} \cdot \text{cm}^2$



Translational degree



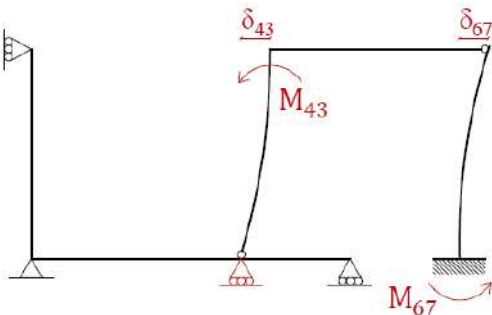
$$d = 2n - b - r = 2 \cdot 7 - 6 - 6 = 2$$

$$n = 7 ; b = 6 ; r = 6$$

Restrained frame FEM

$$M_{47} = \frac{ql^2}{8} = \frac{3000 \cdot 6^2}{8} = 13500 \text{ kg} \cdot \text{m}$$

Displacement 1 FEM

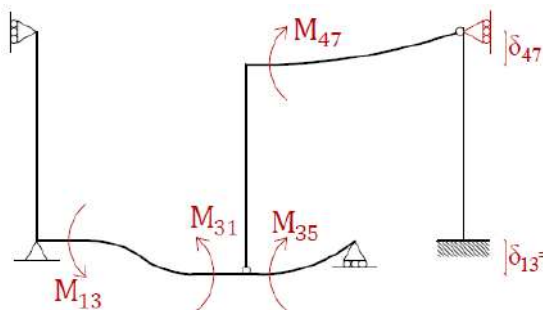


$$\delta_{43} = \delta_{67}$$

$$\frac{M_{43} \cdot l_{43}^2}{3EI} = \frac{M_{67} \cdot l_{67}^2}{3EI} = M_{43} = M_{67}$$

$$M_{43} = M_{67} = 1000\alpha_1$$

Displacement 2 FEM



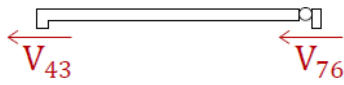
$$\delta_{47} = \delta_{13} = \delta_{35} = \Delta_2$$

$$M_{13} = M_{31} = \frac{\delta_{13} \cdot 6EI}{l_{13}^2} = 0,16 \Delta_2 EI = 500\alpha_2$$

$$M_{47} = -\frac{\delta_{47} \cdot 3EI}{l_{47}^2} = -0,083 \Delta_2 EI = -250\alpha_2$$

$$M_{35} = -\frac{\delta_{35} \cdot 3EI}{l_{35}^2} = -0,3 \Delta_2 EI = -1000\alpha_2 \rightarrow \Delta_2 = \frac{1000\alpha_2}{0,3EI} = \frac{3000\alpha_2}{EI}$$

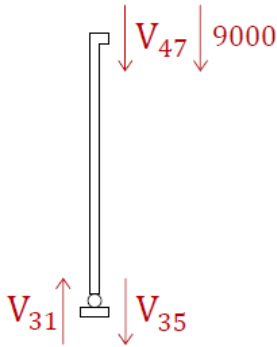
1. displacement equilibrium equation



$$\Sigma H = 0; V_{43} + V_{76} = 0$$

$$\frac{M_{43}}{6} + \frac{M_{76}}{6} = 0; \boxed{M_{43} + M_{76} = 0}$$

2. displacement equilibrium equation



$$\Sigma V = 0; 9000 + V_{47} + V_{35} - V_{31} = 0$$

$$9000 + \frac{M_{47}}{6} + \frac{M_{35}}{3} - \frac{M_{31} + M_{13}}{6} = 0$$

$$\boxed{54000 + M_{47} + 2 M_{35} - M_{31} - M_{13} = 0}$$

Stiffness

$$K_{21} = \frac{3 EI}{l} = \frac{3 EI}{6} = 0,5 EI; K_{13} = \frac{4 EI}{l} = 0,6 EI; K_{35} = \frac{3 EI}{l} = \frac{3 EI}{6} = EI$$

$$K_{34} = \frac{3 EI}{l} = 0,5 EI; K_{47} = \frac{3 EI}{l} = 0,5 EI$$

Distribution factors

1. node

$$r_{12} = \frac{K_{21}}{K_{21} + K_{13}} = \frac{0,5 EI}{(0,5 + 0,6)EI} = 0,43; r_{13} = 1 - r_{12} = 0,57$$

3. node

$$r_{31} = \frac{K_{13}}{K_{13} + K_{35}} = \frac{0,6 EI}{(0,6 + 1)EI} = 0,4; r_{35} = 1 - r_{31} = 0,6$$

4. node

$$r_{43} = \frac{K_{34}}{K_{34} + K_{47}} = \frac{0,5 EI}{(0,5 + 0,5)EI} = 0,5; r_{47} = 1 - r_{43} = 0,5$$

1.STATE (RESTRAINED)		2	1			3			4			7			5	6
	21		12	13		31	34	35		43	47		74	76	53	67
	-		0.43	0.57		0.40	-	0.60		0.50	0.50		-	-	-	-
EFM										13500.0	13500.0					
										-13500.0	-6750.0					
	0.00		0.00	0.00		0.00	0.00	0.00		-6750.00	6750.00		0.00	0.00		0.00
2.STATE (1.DISP)		2	1			3			4			7			5	6
	21		12	13		31	34	35		43	47		74	76	53	67
	-		0.43	0.57		0.40	-	0.60		0.50	0.50		-	-	-	-
EFM x α1										1000.0	1000.0					1000.0
										-1000.0	-500.0					
	0.00		0.00	0.00		0.00	0.00	0.00		500.00	-500.00		0.00	0.00		1000.00
2.STATE (2.DISP)		2	1			3			4			7			5	6
	21		12	13		31	34	35		43	47		74	76	53	67
	-		0.43	0.57		0.40	-	0.60		0.50	0.50		-	-	-	-
EFM x α2		500.0		500.0						250.0	-250.0					
		-500.0	-214.3	-285.7	-642.9	500.0	-	-1000.0								
		128.6	128.6	128.6	642.9	257.1		385.7		250.0	125.0	125.0				
		-128.6	-55.1	-73.5	-36.7	-36.7										
		7.3	7.3	36.7	14.7	14.7		22.0								
		-7.3	-3.1	-4.2	-2.1	-2.1										
		0.4	0.4	2.1	0.8	0.8		1.3								
		-0.4	-0.2	-0.2	-0.1	-0.1										
					0.1	0.0		0.1								
		0.0	-272.7	272.7		590.9	0.0	-590.9		125.0	-125.0		0.0	0.0		0.0
		2	1			3			4			7			5	6
	21		12	13		31	34	35		43	47		74	76	53	67
1.STATE	0		0	0		0	0	0		-6750	6750		0	0	0	0
2.STATE α1 ;	0	0	0	0		0	0	0		1105	-1105		0	0	0	2210
3.STATE α2;	0	-7494	7494		16239	0	-16239		3435	-3435		0	0	0	0	0
FINAL RESULTS	0	-7494	7494		16239	0	-16239		-2210	2210		0	0	0	0	2210

Equation resolution

1) $M_{43} + M_{76} = 0$; $\boxed{-6750 + 1500\alpha_1 + 125\alpha_2 = 0}$

$$M_{43} = -6750 + 500\alpha_1 + 125\alpha_2$$

$$M_{76} = 1000\alpha_1$$

2) $54000 + M_{47} + 2 M_{35} - M_{31} - M_{13} = 0$; $\boxed{60750 - 500\alpha_1 - 2174,5\alpha_2 = 0}$

$$M_{47} = 6750 - 500\alpha_1 - 125\alpha_2$$

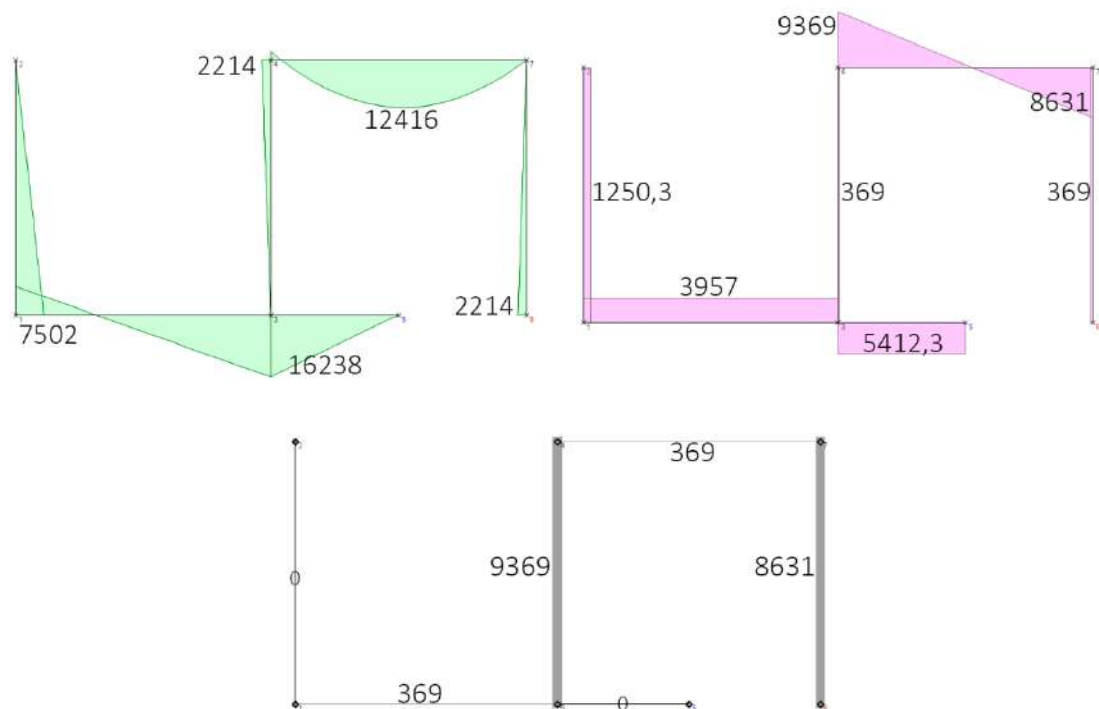
$$M_{35} = -592\alpha_2$$

$$M_{31} = 592\alpha_2$$

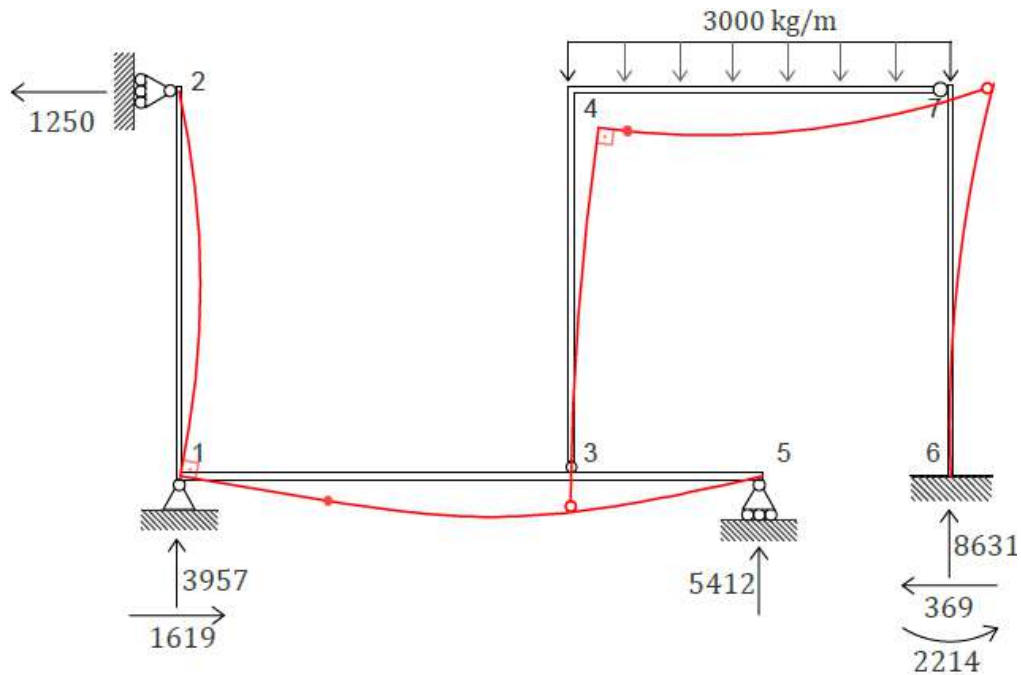
$$M_{13} = 273,5\alpha_2$$

$$\boxed{\alpha_1 = 2,2143 ; \alpha_2 = 27,4283}$$

Diagrams



Reactions and deformation



$$\Sigma H = 0 ; 1619 = 369 + 1250 ; 1619 = 1619 \checkmark$$

$$\Sigma V = 0 ; 18000 = 3957 + 5412 + 8631 ; 18000 = 18000 \checkmark$$

$$\Sigma M_1 = 0 ;$$

$$3000 \cdot 6 \cdot 9 - 5412 \cdot 9 - 8631 \cdot 12 - 2214 - 1250 \cdot 6 = 0 ;$$

$$162000 - 48708 - 103572 - 2214 - 7500 = 0 ; 6 \cong 0 \checkmark$$

7. node horizontal displacement (in 7-6 column)

$$\delta_7 = \frac{M_{76} \cdot l_{76}^2}{3EI} = \frac{221400 \cdot 600^2}{3 \cdot 5,284 \cdot 10^6} = 0,5 \text{ cm (eskuinera)}$$

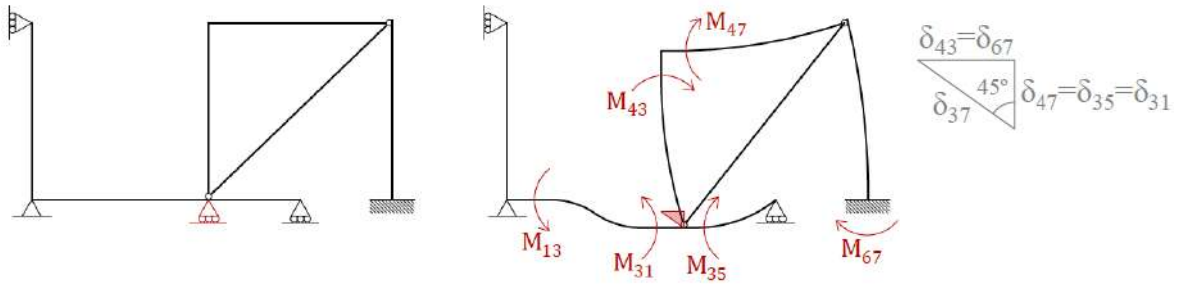
$$M_{76} = 1000\alpha_1 = 1000 \cdot 2,214 = 2214 \text{ kg} \cdot \text{m} = 221400 \text{ kg} \cdot \text{cm}$$

3. node vertical displacement (in 3-5 column)

$$\delta_3 = \frac{M_{35} \cdot l_{35}^2}{3EI} = \frac{-2742830 \cdot 300^2}{3 \cdot 5,284 \cdot 10^6} = 1,56 \text{ cm (behera)}$$

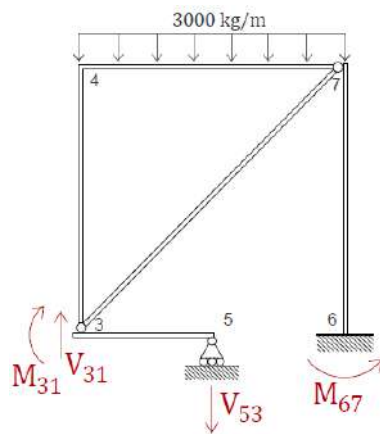
$$M_{35} = -1000\alpha_2 = -1000 \cdot 27,4283 = -27428,3 \text{ kg} \cdot \text{m} = -2742830 \text{ kg} \cdot \text{cm}$$

With the diagonal bar



$$M_{35} = -1000\alpha ; M_{31} = 500\alpha ; M_{47} = M_{67} = M_{43} = -250\alpha$$

Equilibrium equation



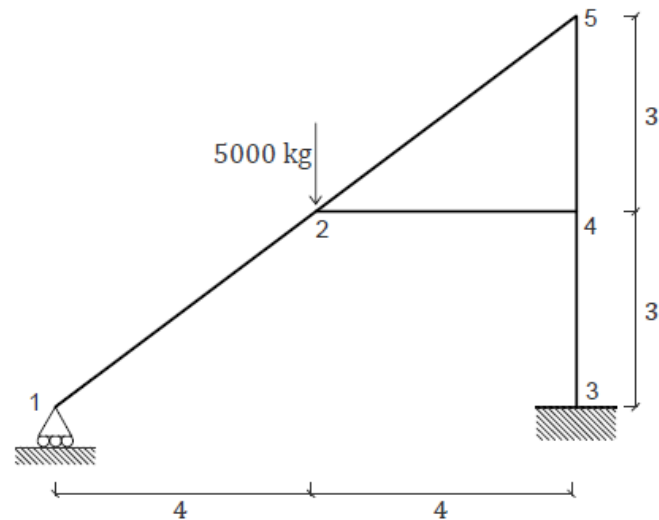
$$\Sigma M_6 = 0;$$

$$-3000 \cdot 6 \cdot 3 + M_{31} + 6 \cdot V_{31} - 3 \cdot V_{53} - M_{67} = 0$$

$$-54000 + M_{31} + 6 \frac{M_{31} + M_{13}}{6} - 3 \frac{M_{35}}{3} - M_{67} = 0$$

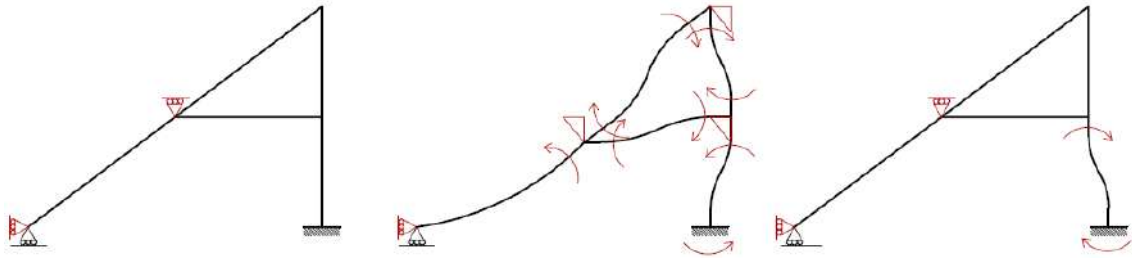
$$\boxed{-54000 + 2 M_{31} + M_{13} - M_{35} - M_{67} = 0}$$

5.2.11. Sway frames with roller supports. Example



- Using Cross method, draw the bending moment, shear and axial strength diagrams.
- Draw the deformation.
- Calculate 2 node's displacement.

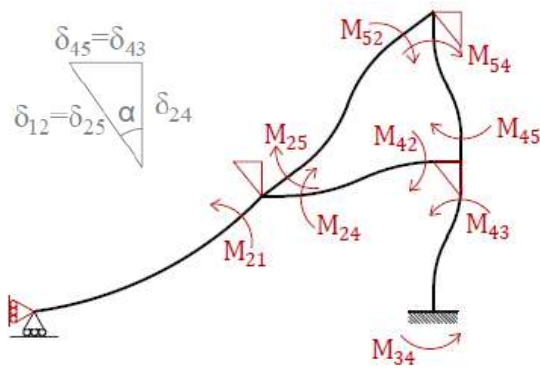
Translational degree



$$d = 2n - b - r = 2 \cdot 5 - 5 - 3 = 2$$

$$n = 5 ; b = 5 ; r = 3$$

1. Displacement



$$\sin \alpha = 0,6 ; \cos \alpha = 0,8$$

$$\delta_{12} = \delta_{25} = \Delta_1 ; \delta_{45} = \delta_{43} = 0,6 \cdot \Delta_1 ; \delta_{24} = 0,8 \cdot \Delta_1$$

$$M_{21} = \delta_{12} \frac{3EI}{l_{12}^2} = 0,12 \Delta_1 EI = 300\alpha_1$$

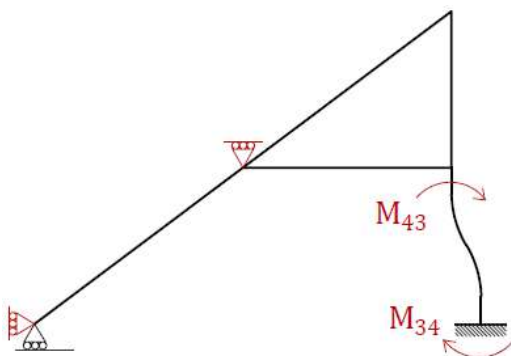
$$M_{25} = M_{52} = -\delta_{25} \frac{6EI}{l_{25}^2} = -0,24 \Delta_1 EI = -600\alpha_1$$

$$M_{24} = M_{42} = -\delta_{24} \frac{6EI}{l_{24}^2} = -0,3 \Delta_1 EI = -750\alpha_1$$

$$M_{45} = M_{54} = -\delta_{45} \frac{6EI}{l_{45}^2} = -0,4 \Delta_1 EI = -1000\alpha_1$$

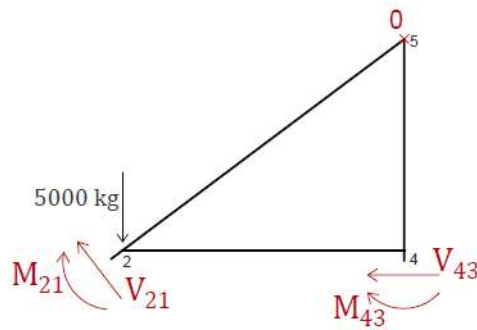
$$M_{34} = M_{43} = \delta_{34} \frac{6EI}{l_{34}^2} = 0,4 \Delta_1 EI = 1000\alpha_1 \rightarrow \Delta_1 = \frac{2500\alpha_1}{EI}$$

2. Displacement



$$M_{34} = M_{43} = -1000\alpha_2$$

1. displacement balance equation



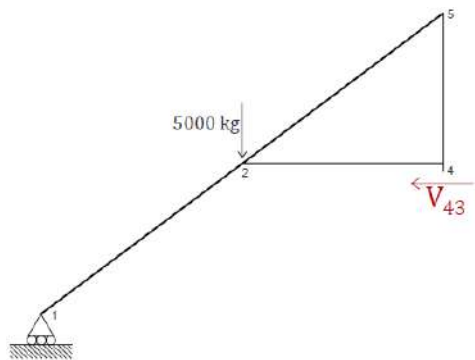
$$\Sigma M_0 = 0$$

$$M_{21} + M_{43} + 5 \cdot V_{21} + 3 \cdot V_{43} = 20000$$

$$M_{21} + M_{43} + 5 \frac{M_{21}}{5} + 3 \frac{M_{43} + M_{34}}{3} = 20000$$

$$2 M_{21} + 2 M_{43} + M_{34} = 0$$

2. displacement balance equation



$$\Sigma H = 0 ; V_{43} = 0$$

$$\frac{M_{43} + M_{34}}{3} = 0$$

$$M_{43} + M_{34} = 0$$

Stiffness

$$K_{21} = \frac{3EI}{l} = 0,6 EI ; K_{24} = \frac{4EI}{l} = EI ; K_{25} = \frac{4EI}{l} = 0,8 EI ; K_{43} = K_{45} = \frac{4EI}{l} = 1, \hat{3} EI$$

Distribution factors

2. node

$$r_{21} = \frac{K_{21}}{K_{21} + K_{24} + K_{25}} = \frac{0,6 EI}{2,4 EI} = 0,25 ; r_{24} = \frac{EI}{2,4 EI} = 0,41\hat{6} ; r_{25} = 1 - r_{21} - r_{24} = 0, \hat{3}$$

4. node

$$r_{43} = \frac{K_{43}}{K_{43} + K_{45} + K_{42}} = \frac{1, \hat{3} EI}{3, \hat{6} EI} = 0, \hat{3} \hat{6} ; r_{45} = r_{43} = 0, \hat{3} \hat{6} ; r_{42} = 1 - r_{43} - r_{45} = 0, \hat{2} \hat{7}$$

5. node

$$r = \frac{K_{25}}{K_{25} + K_{45}} = \frac{0,8 EI}{2,1 \hat{3} EI} = 0,375 ; r_{53} = 1 - r_{52} = 0,625$$

1.STATE (1.DISP)					1			2			4			5			3
					12		21	25	24		42	43	45		54	52	34
EFM x α1					-		0,250	0,333	0,417		0,273	0,364	0,364		0,625	0,375	-
							300,0	-600,0	-750,0		-750,0	1000,0	-1000,0		-1000,0	-600,0	1000,0
						-1050,0											
						1050,0	262,5	350,0	437,5	-531,3	218,8					175,0	
									72,4	531,3	144,9	193,2	193,2	-1328,4	96,6		96,6
						321,5		249,1						415,1	1328,4	830,3	498,2
						-321,5	-80,4	-107,2	-134,0	348,1	-67,0					-53,6	
									-47,5	-348,1	-94,9	-126,6	-126,6	-116,9	-63,3		-63,3
						-25,6		21,9					36,5	116,9	73,1	43,8	
						25,6	6,4	8,5	10,6	41,9	5,3					4,3	
									-5,7	-41,9	-11,4	-15,2	-15,2	-3,3	-7,6		-7,6
						-5,1		0,6					1,0	3,3	2,1	1,3	
						5,1	1,3	1,7	2,1	2,1	1,1					0,8	
										-2,1	-0,6	-0,8	-0,8	0,5	-0,4		-0,4
														-0,5	-0,3	-0,2	
					0,00	489,78	-75,34	-414,44		-553,90	1050,60	-496,70		-69,59	69,59	1025,30	
2.STATE (2.DISP)					1	2			4			5			3		
					12		21	25	24		42	43	45		54	52	34
EFM x α2					-		0,250	0,333	0,417		0,273	0,364	0,364		0,625	0,375	-
						0,0											
						0,0	0,0	0,0	0,0	-1000,0	0,0					0,0	
									136,4	1000,0	272,7	363,6	363,6	181,8	181,8		181,8
						102,3		-34,1					-56,8	-181,8	-113,6	-68,2	
						-102,3	-25,6	-34,1	-42,6	-78,1	-21,3					-17,0	
									10,7	78,1	21,3	28,4	28,4	-2,8	14,2		14,2
						11,2		0,5					0,9	2,8	1,8	1,1	
						-11,2	-2,8	-3,7	-4,7	-1,4	-2,3					-1,9	
									0,2	1,4	0,4	0,5	0,5	-1,6	0,3		0,3
						0,5		0,3					0,5	1,6	1,0	0,6	
						-0,5	-0,1	-0,2	-0,2	0,4	-0,1					-0,1	
										-0,4	-0,1	-0,1	-0,1	-0,2	-0,1		-0,1
															0,2	0,1	0,1
					0,00	-28,49	-71,24	99,73		270,58	-607,57	337,00		85,45	-85,45	-803,79	
					1	2			4			5			3		
					12		21	25	24		42	43	45		54	52	34
1. STATExα1: α1= 18,999					0,00		9305,18	-1431,37	-7873,82		-10523,41	19960,05	-9436,65		-1322,07	1322,07	19479,39
2. STATExα2: α2= 27,944					0,00		-796,10	-1990,84	2786,94		7561,11	-16978,21	9417,10		2387,84	-2387,84	-22461,23
FINAL RESULTS					0,00		8509,08	-3422,20	-5086,88		-2962,30	2981,84	-19,55		1065,77	-1065,77	-2981,84

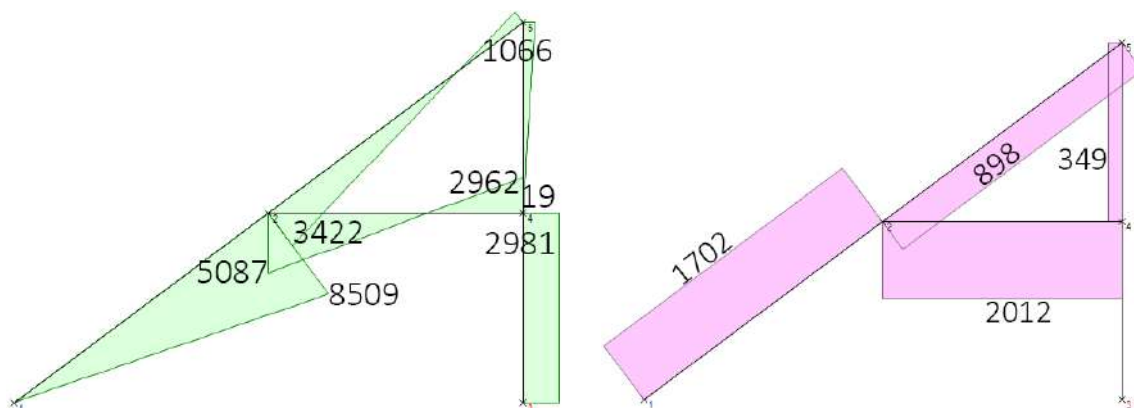
Equation resolution

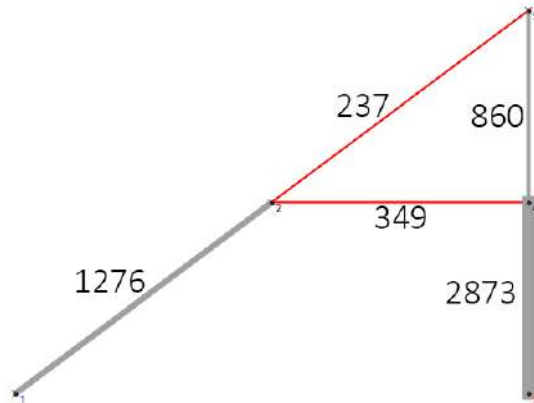
$$1) 2 M_{21} + 2 M_{43} + M_{34} = 0 ; 4106,06\alpha_1 - 2075,91\alpha_2 = 20000$$

$$2) M_{43} + M_{34} = 0 ; 2075,9\alpha_1 - 411,36\alpha_2 = 0$$

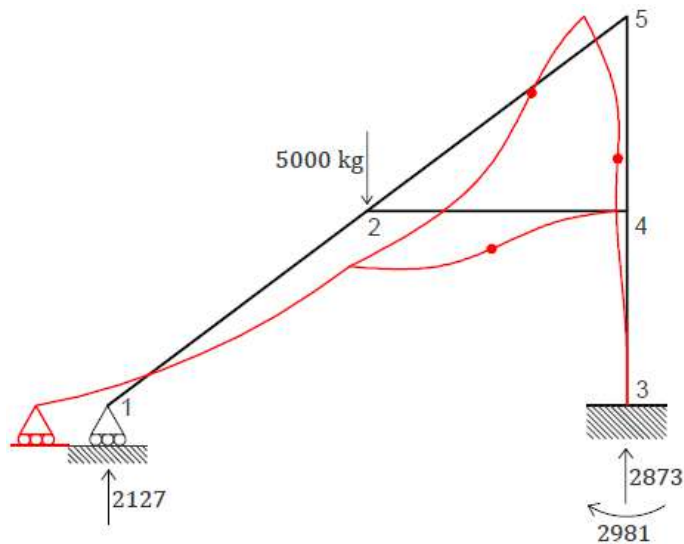
$$\alpha_1 = 18,999 ; \alpha_2 = 27,944$$

Diagrams





Reactions and deformation



2. joint displacement

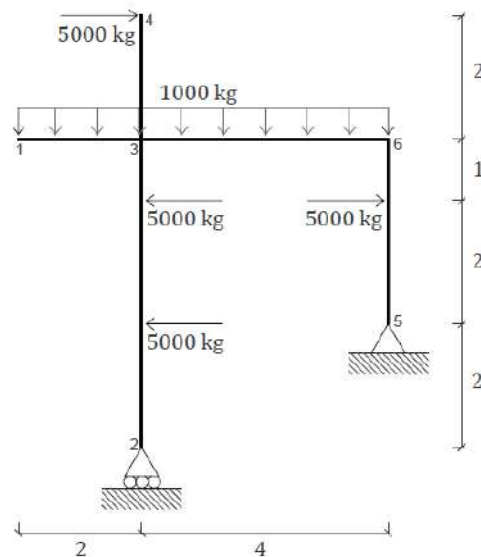
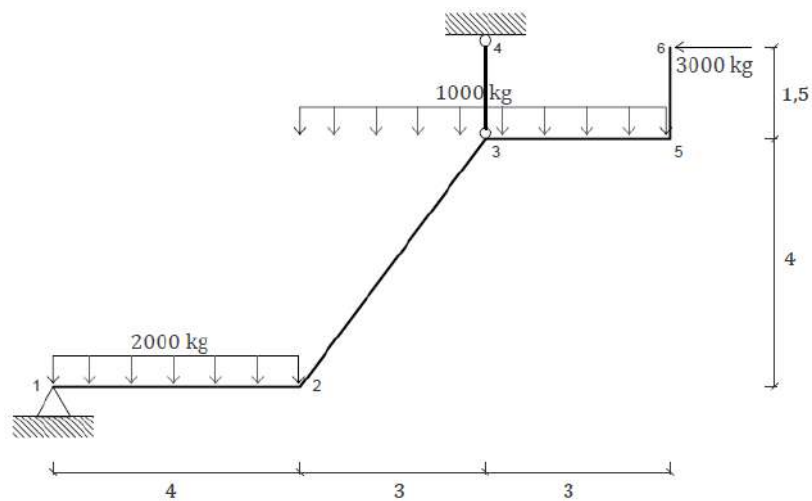
$$\delta_H = \frac{1000\alpha_1 \cdot l_{43}^2}{6EI} - \frac{1000\alpha_2 \cdot l_{43}^2}{6EI} = \frac{(18999 - 27944)100 \cdot 300^2}{6 \cdot 2,1 \cdot 10^{10}} = -0,64 \text{ cm (ezkerrera)}$$

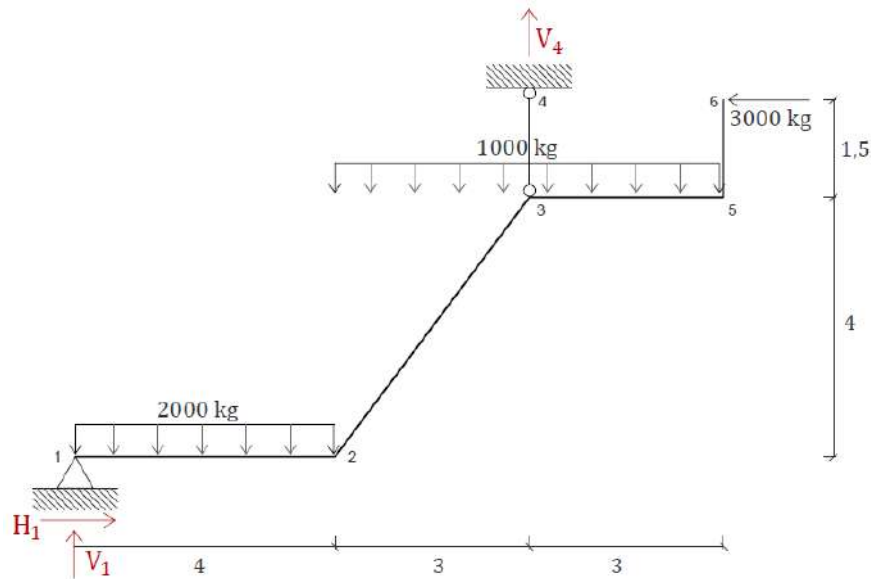
$$\delta_V = \frac{750\alpha_1 \cdot l_{24}^2}{6EI} = \frac{750 \cdot 18,999 \cdot 100 \cdot 400^2}{6 \cdot 2,1 \cdot 10^{10}} = 1,81 \text{ cm (behera)}$$

6. EXAM EXERCISES

6.1. ISOSTATIC FRAMES. REACTIONS AND DIAGRAMMS.

Solve the structure's reactions and draw the bending moment, shear and axial strength diagrams. Draw the deformation.



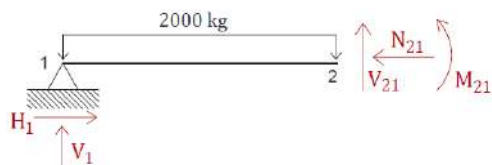


$$\Sigma H = 0 ; \boxed{H_1 = 3000 \text{ kg}}$$

$$\Sigma V = 0 ; 2000 \cdot 4 + 1000 \cdot 6 = V_1 + V_4 ; \boxed{V_1 = 14000 - 5928,6 = 8071,4 \text{ kg}}$$

$$\Sigma M_1 = 0 ; 2000 \cdot 4 \cdot 2 + 1000 \cdot 6 \cdot 7 - 3000 \cdot 5,5 = V_4 \cdot 7 ; \boxed{V_4 = 5928,6 \text{ kg}}$$

1-2 section cut

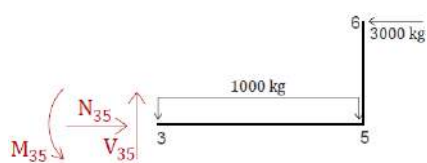


$$H_1 = N_{21} ; N_{21} = 3000 \text{ kg}$$

$$V_1 + V_{21} = 2000 \cdot 4 ; V_{21} = -71,4 \text{ kg}$$

$$M_{21} = V_1 \cdot 4 - 2000 \cdot 4 \cdot 2 = 32285,6 - 16000 = 16285,6 \text{ kg}$$

3-5 section cut

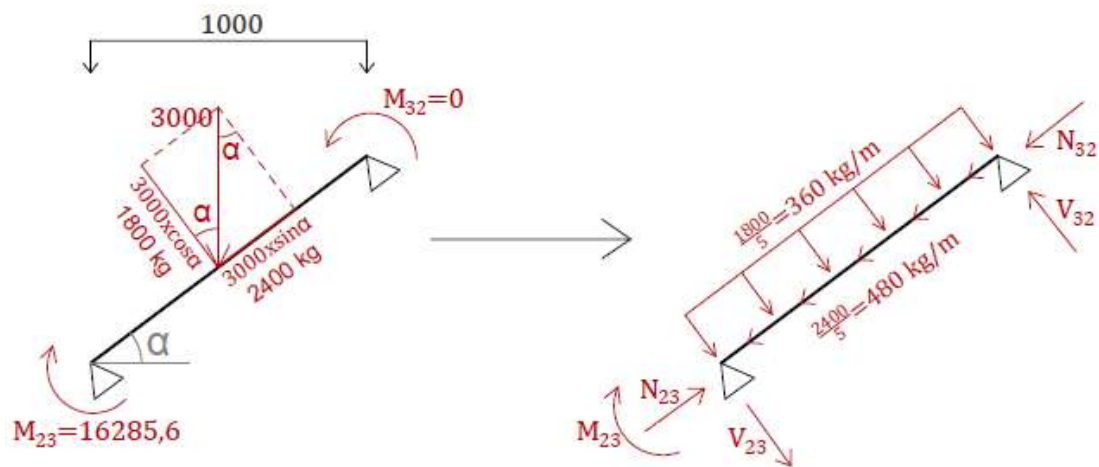


$$N_{35} = 3000 \text{ kg}$$

$$V_{35} = 1000 \cdot 3 = 3000$$

$$M_{35} = 1000 \cdot 3 \cdot 1,5 - 3000 \cdot 1,5 = 0$$

1-2 bar analyse. Shear and axial strengths

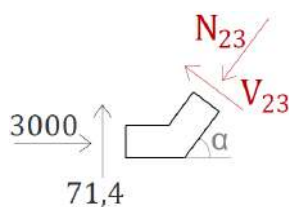


$$\sin \alpha = \frac{4}{5} = 0,8 ; \cos \alpha = \frac{3}{5} = 0,6$$

$$\Sigma V = 0 ; V_{32} = 360 \cdot 5 + 2357 = 4157 \text{ kg}$$

$$\Sigma H = 0 ; N_{32} = N_{23} - 480 \cdot 5 ; N_{23} = 1857 - 2400 = -543 \text{ kg}$$

Balance on 2. joint



$$\Sigma H = 0 ; 3000 = V_{23} \cdot \sin \alpha + N_{23} \cdot \cos \alpha$$

$$3000 = 0,8 \cdot V_{23} + 0,6 \cdot N_{23}$$

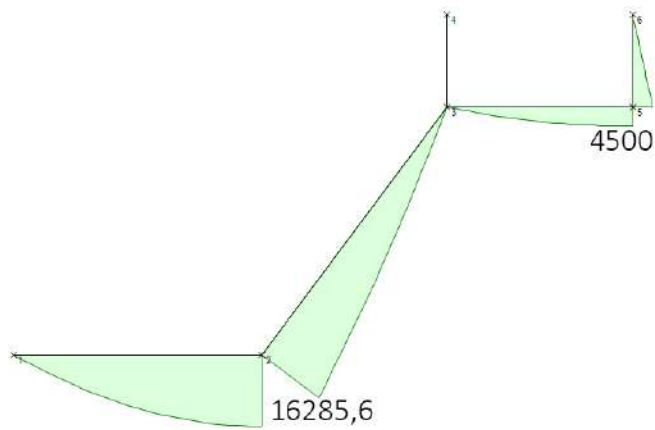
$$V_{23} = 3750 - 0,75 \cdot N_{23} = 2357 \text{ kg}$$

$$\Sigma V = 0 ; N_{23} \cdot \sin \alpha - V_{23} \cdot \cos \alpha = 71,4 ; N_{23} \cdot 0,8 - V_{23} \cdot 0,6 = 71,4$$

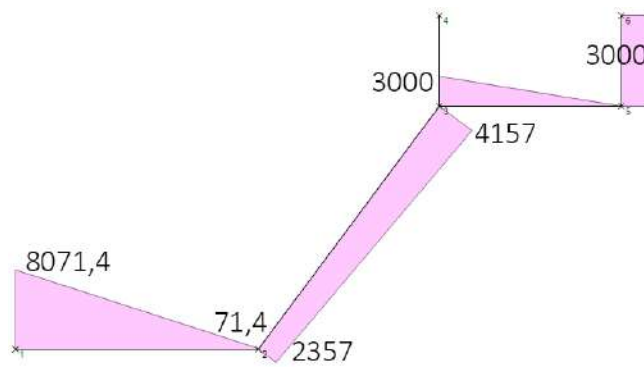
$$0,8 \cdot N_{23} - 0,6 (3750 - 0,75 \cdot N_{23}) = 71,4 ; 0,8 N_{23} - 2250 + 0,45 N_{23} = 71,4$$

$$N_{23} = \frac{2321,4}{1,25} = 1857 \text{ kg}$$

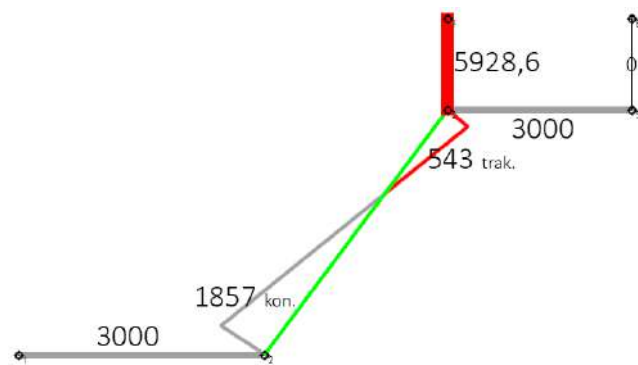
Diagrams



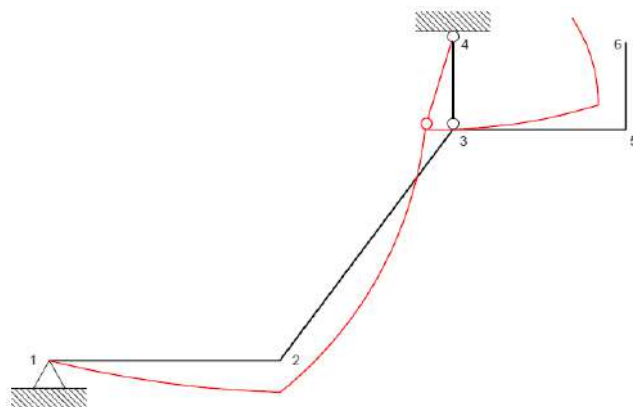
Bending moment



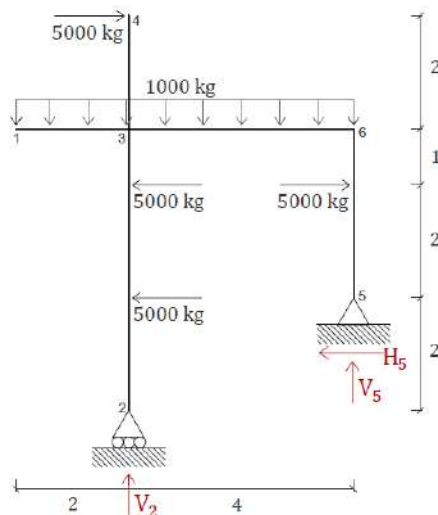
Shear



Axial



Deformation



$$\Sigma H = 0 ; 5000 + 5000 = 5000 + 5000 + H_5$$

$$H_5 = 0$$

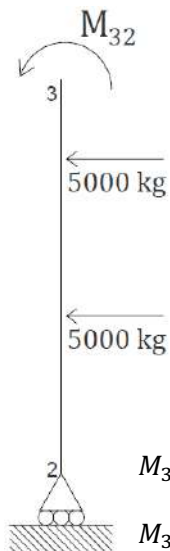
$$\Sigma V = 0 ; V_2 + V_5 = 1000 \cdot 6 \rightarrow$$

$$V_2 = 6000 - 7750 = -1750 \text{ kg}$$

$$\Sigma M_2 = 0 ; 5000 \cdot 2 + 5000 \cdot 4 + V_5 \cdot 4 - 5000 \cdot 4 - 5000 \cdot 7 - 1000 \cdot 6 \cdot 1 = 0$$

$$V_5 = \frac{31000}{4} = 7750 \text{ kg}$$

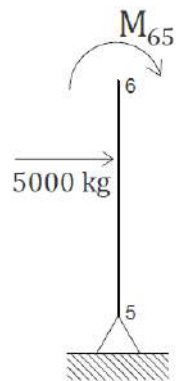
2-3 bar



$$M_{32} = 5000 \cdot 1 + 5000 \cdot 3$$

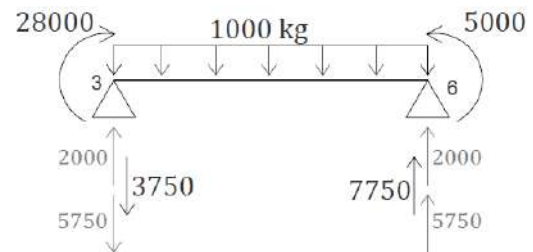
$$M_{32} = 20000 \text{ kg} \cdot \text{m}$$

5-6 bar

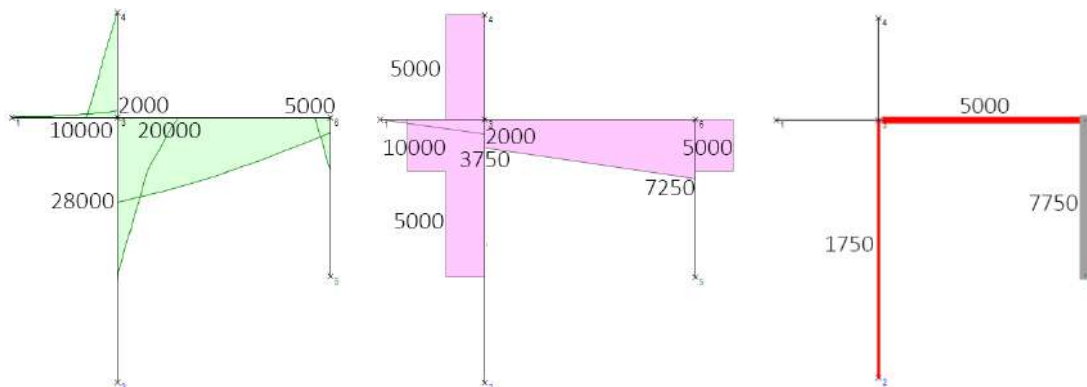


$$M_{32} = 5000 \text{ kg} \cdot \text{m}$$

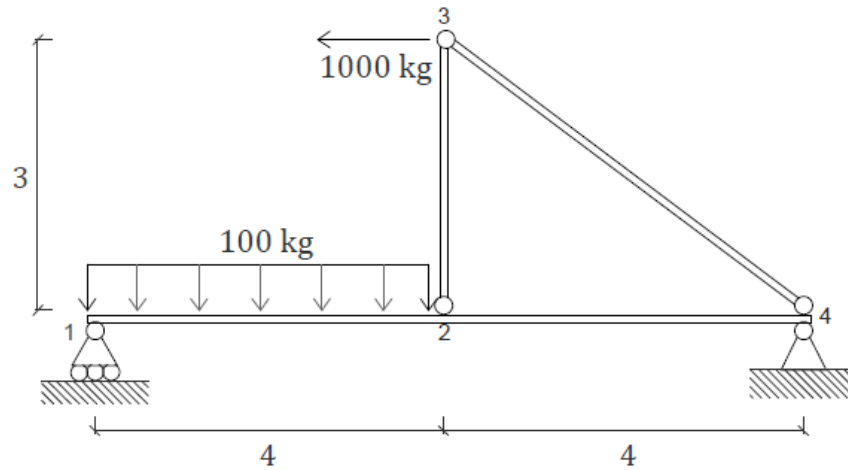
3-6 bar



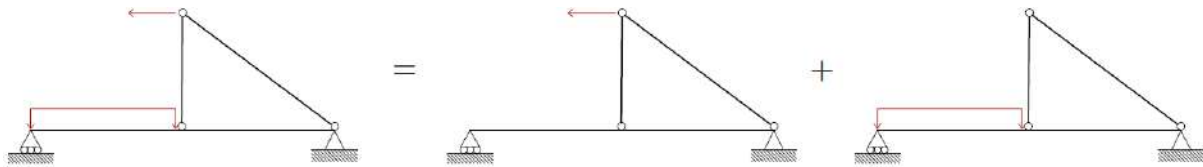
Diagrams



6.2. ISOSTATIC FRAMES. UNIT LOAD METHOD

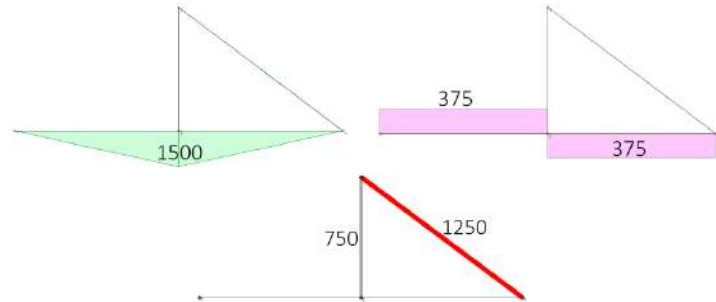
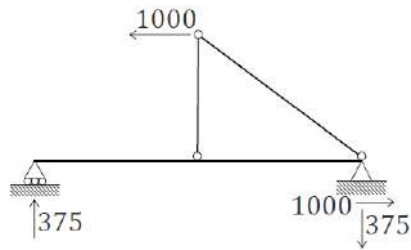


- Using the unit load method, calculate the displacement of the 2 node
- Calculate the rotation of the 1. and 4. supports
- Draw the bending moment, shear and axial strength diagrams.

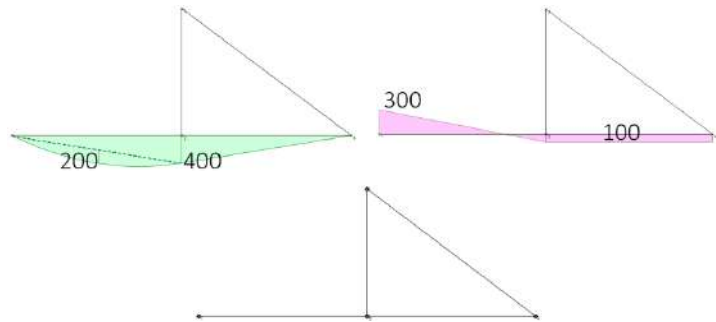
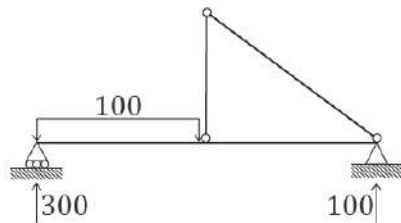


Real system

a)

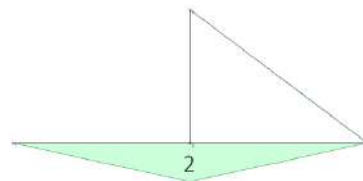
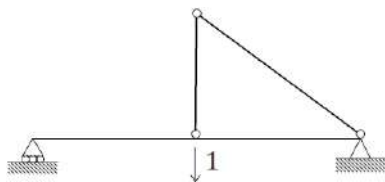


b)

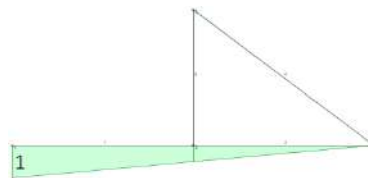
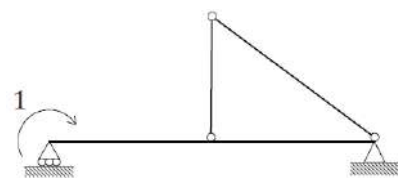


Virtual system

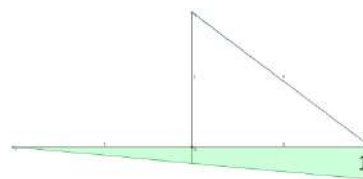
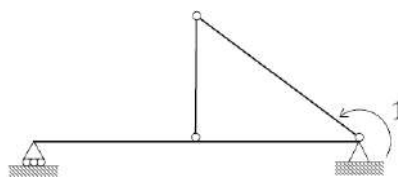
1)



2)



3)



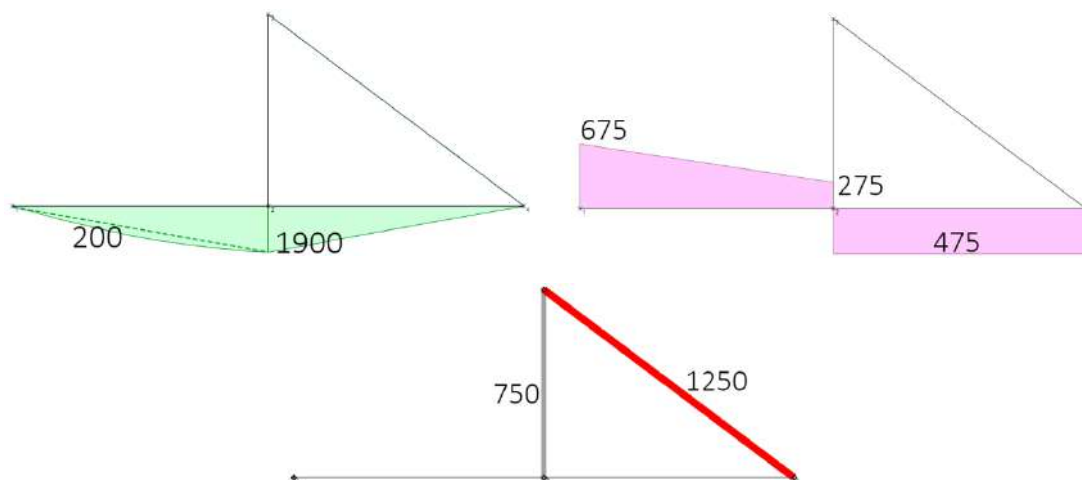
R U	
	$\delta = \left[\left(\frac{1}{3} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l \right) \right] \cdot \frac{1}{EI}$
	$\sigma_1 = \left[\left(\frac{1}{4} M \cdot M' \cdot l \right) + \left(\frac{1}{4} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l \right) \right] \cdot \frac{1}{EI}$
	$\sigma_4 = \left[\left(\frac{1}{4} M \cdot M' \cdot l \right) + \left(\frac{1}{4} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l \right) \right] \cdot \frac{1}{EI}$

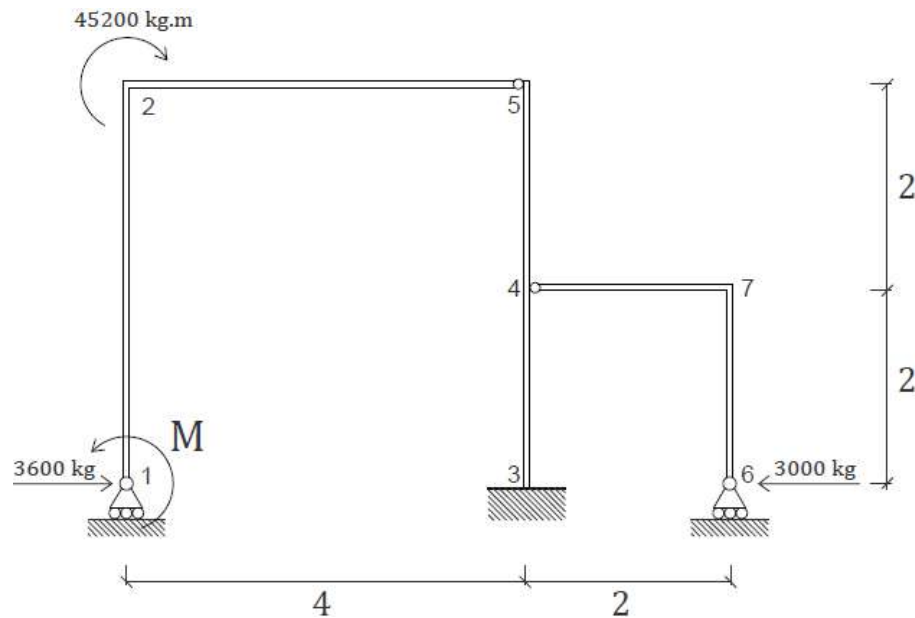
$$\delta = \left(\frac{1}{3} 2 \cdot 1500 \cdot 8 + \frac{1}{3} 2 \cdot 400 \cdot 8 + \frac{1}{3} 2 \cdot 200 \cdot 4 \right) \cdot \frac{1}{EI} = \frac{10666,6}{8,169 \cdot 10^5} = 0,013 = \boxed{1,3 \text{ cm}}$$

$$\sigma_1 = \left(\frac{1}{4} 1 \cdot 1500 \cdot 8 + \frac{1}{4} 1 \cdot 400 \cdot 8 + \frac{1}{3} 1 \cdot 200 \cdot 4 + \frac{1}{3} 0,5 \cdot 200 \cdot 4 \right) \cdot \frac{1}{EI} = \frac{4200}{8,169 \cdot 10^5} = \boxed{5,1 \cdot 10^{-3} \text{ Rad}}$$

$$\sigma_4 = \left(\frac{1}{4} 1 \cdot 1500 \cdot 8 + \frac{1}{4} 1 \cdot 400 \cdot 8 + \frac{1}{3} 0,5 \cdot 200 \cdot 4 \right) \cdot \frac{1}{EI} = \frac{3933,3}{8,169 \cdot 10^5} = \boxed{4,81 \cdot 10^{-3} \text{ Rad}}$$

Diagrams

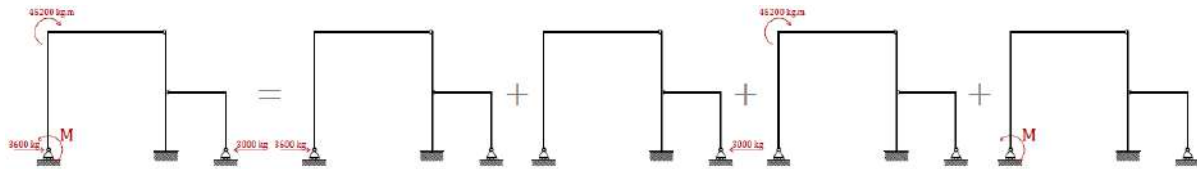




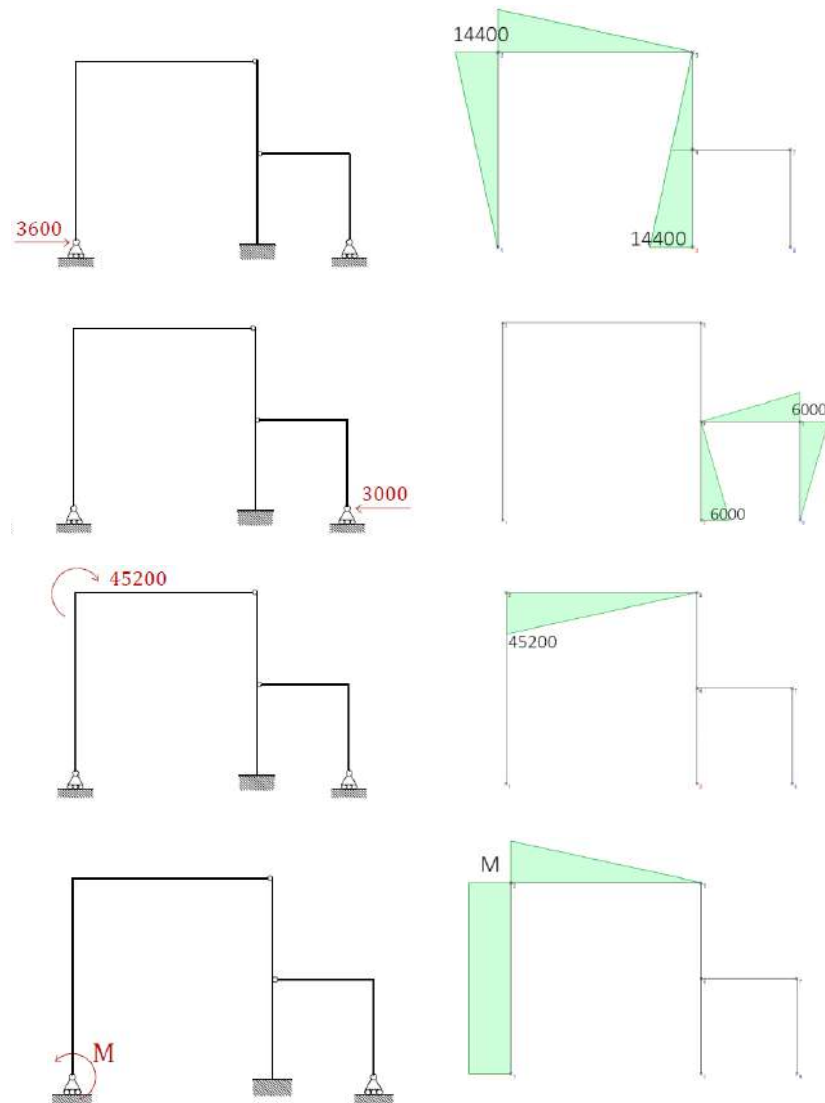
- Using the unit load method, calculate what the value of M must be in order for the rotation of that support to be zero.
- Draw the bending moment, shear and axial strength diagrams.
- Calculate the displacement of the 1 and 6 supports.
- Draw the deformation.

Data:

$$EI = \text{constant} = 6,25 \cdot 10^{10} \text{ kg} \cdot \text{cm}^2$$



Real system:



Virtual system:

R U	
	$\vartheta = \left[\frac{1}{2} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l - \frac{1}{3} M \cdot M' \cdot l + M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l \right] \cdot \frac{1}{EI}$

$$\vartheta = \left(\frac{1}{2} 1 \cdot 14400 \cdot 4 + \frac{1}{3} 1 \cdot 14400 \cdot 4 - \frac{1}{3} 1 \cdot 45200 \cdot 4 + 1 \cdot M \cdot 4 + \frac{1}{3} 1 \cdot M \cdot 4 \right) \cdot \frac{1}{EI}$$

$$28800 + 19200 - 60266,6 + 5,3M = 0 ; \boxed{M = 2300 \text{ kg} \cdot \text{m}}$$

Displacement of 1. and 6. supports

R U	
	$\delta_1 = \frac{1}{EI} \left[\left(\frac{1}{3} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l \right) + \left(\frac{-1}{3} M \cdot M' \cdot l - \frac{1}{6} M \cdot M' \cdot l \right) + \left(\frac{-1}{3} M \cdot M' \cdot l \right) + \left(\frac{1}{2} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l \right) \right]$
	$\delta_1 = \frac{1}{EI} \left[\left(\frac{-1}{3} M \cdot M' \cdot l - \frac{1}{6} M \cdot M' \cdot l \right) + \left(\frac{1}{3} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l + \frac{1}{3} M \cdot M' \cdot l \right) \right]$

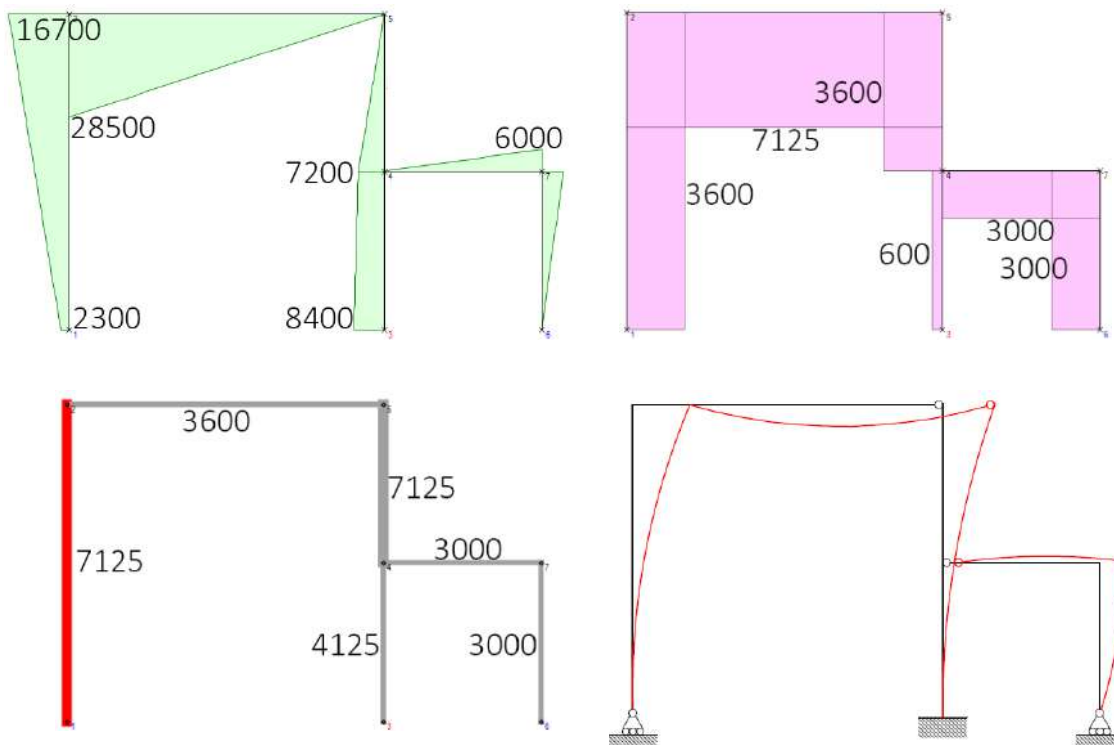
$$\delta_1 = \frac{1}{EI} \left[\left(\frac{1}{3} 4 \cdot 14400 \cdot 4 + \frac{1}{3} 4 \cdot 14400 \cdot 4 + \frac{1}{3} 4 \cdot 14400 \cdot 4 \right) + \left(\frac{-1}{3} 4 \cdot 6000 \cdot 2 - \frac{1}{6} 2 \cdot 6000 \cdot 2 \right) + \left(\frac{-1}{3} 4 \cdot 45200 \cdot 4 \right) + \left(\frac{1}{2} 4 \cdot 2300 \cdot 4 + \frac{1}{3} 4 \cdot 2300 \cdot 4 \right) \right]$$

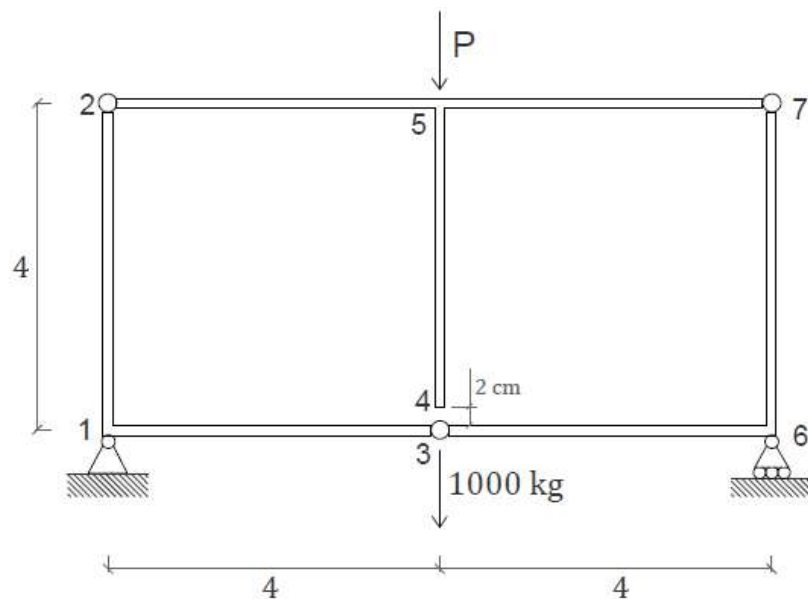
$$\delta_1 = \frac{1}{EI} [3 \cdot 76800 - 16000 - 4000 - 241066,6 + 18400 + 12266,6] ; \boxed{\delta_1 = 0}$$

$$\delta_6 = \frac{1}{EI} \left[\left(\frac{-1}{3} 2 \cdot 14400 \cdot 2 - \frac{1}{6} 2 \cdot 7200 \cdot 2 \right) + \left(\frac{1}{3} 2 \cdot 6000 \cdot 2 + \frac{1}{3} 2 \cdot 6000 \cdot 2 + \frac{1}{3} 2 \cdot 6000 \cdot 2 \right) \right]$$

$$\delta_6 = \frac{1}{EI} [-19200 - 4800 + 3 \cdot 8000] ; \boxed{\delta_6 = 0}$$

Diagrams and deformations





- Using the Unit Load method, calculate what the value of P load must be in order for the 4 and 3 nodes to touch each other.
- With calculated P load, draw the bending moment, shear and axial strength diagrams and draw the deformation.

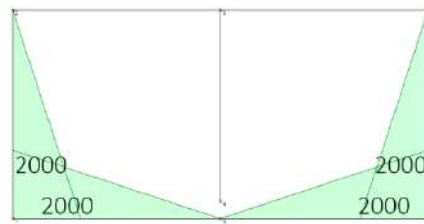
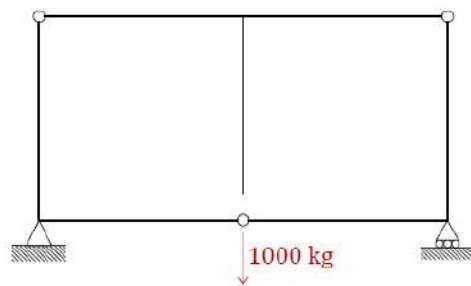
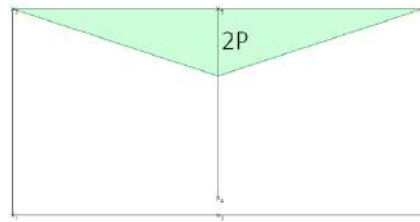
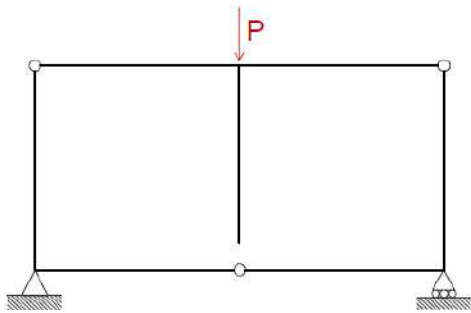
Data:

$$EI = \text{constant}$$

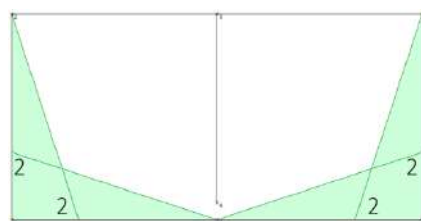
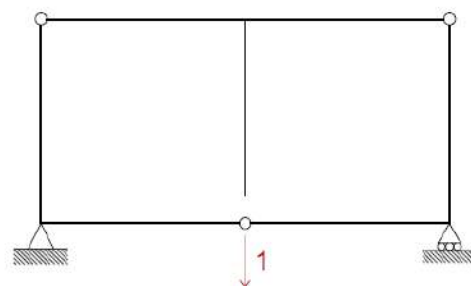
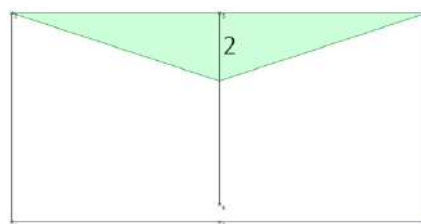
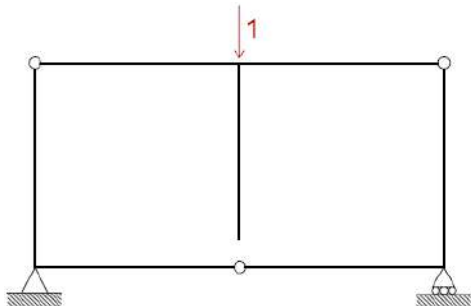
$$E = 2 \cdot 10^5 \text{ kg/cm}^2$$

$$I = 8 \cdot 10^4 \text{ cm}^4$$

Real system:



Virtual systems



3. joint displacement

	$\delta_3 = \frac{1}{EI} \left(\frac{1}{3} \cdot 2 \cdot 2000 \cdot 4 \right) \cdot 4 = \frac{21333, \hat{3}}{EI} m$

4. joint displacement

	$\delta_4 = \frac{1}{EI} \left(\frac{1}{3} \cdot 2 \cdot 2 P \cdot 4 \right) \cdot 2 = \frac{P \cdot 10, \hat{6}}{EI} m$

To fulfil the statement condition:

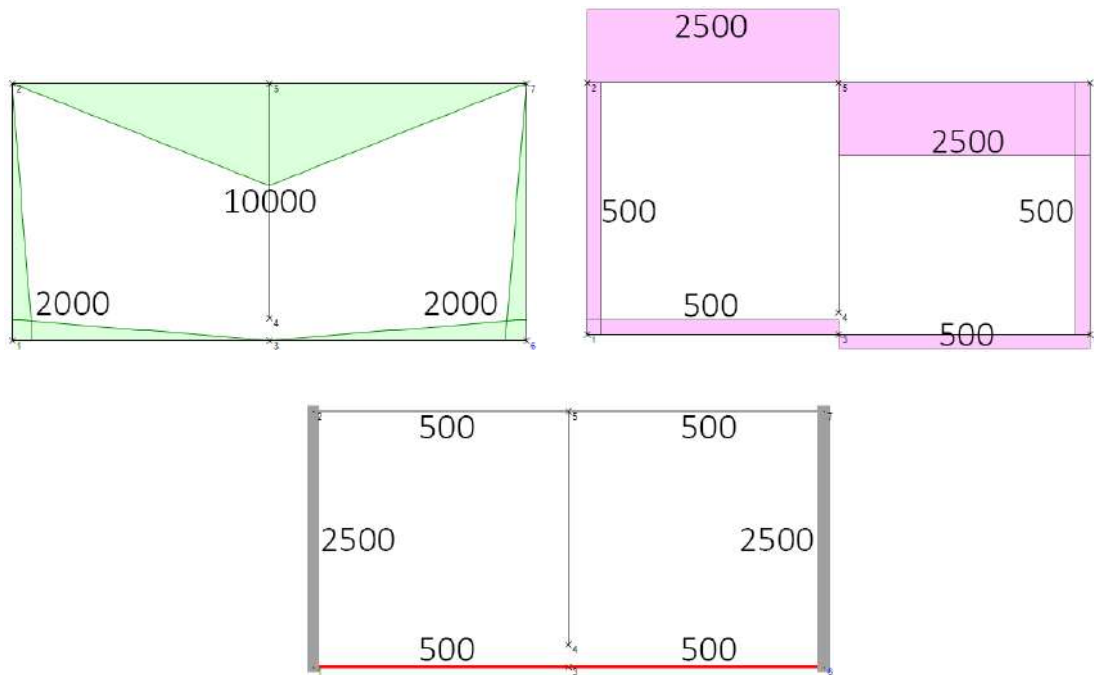
$$\delta_4 = \delta_3 + 0,02 m$$

$$\frac{P \cdot 10, \hat{6}}{EI} = \frac{21333, \hat{3}}{EI} + 0,02 ; P = \frac{EI}{10, \hat{6}} \cdot \left(\frac{21333, \hat{3}}{EI} + 0,02 \right) ; P = 2000 + 1,875 \cdot 10^3 \cdot EI$$

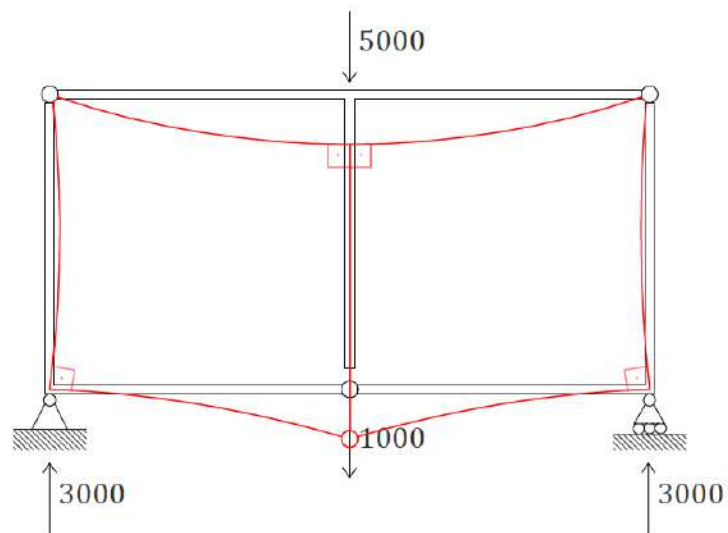
$$EI = 200000 \cdot 80000 = 1,6 \cdot 10^{10} kg \cdot cm^2 = 1,6 \cdot 10^6 kg \cdot m^2$$

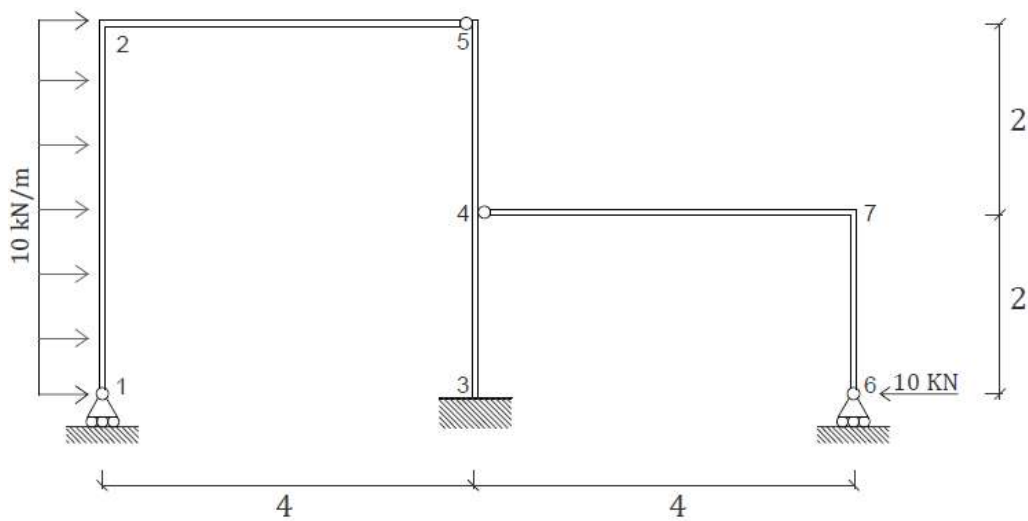
$$P = 2000 + 1,875 \cdot 10^3 \cdot 1,6 \cdot 10^6 = 2000 + 3000 ; \boxed{P = 5000 kg}$$

Diagrams



Reactions deformation





- Using the unit load method, calculate what the value of EI constants must be, in order for the horizontal displacement of the structure to be 2 cm.
- Draw the bending moment, shear and axial strength diagrams.
- Draw the deformation.
- Analyse how the structure would work if the 3 support would be a hinged support.

Data:

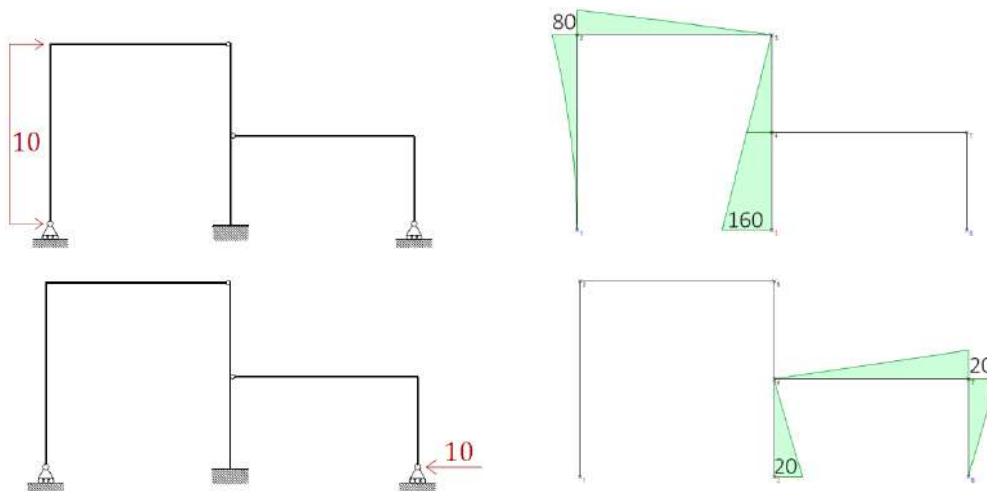
$EI = \text{constant}$

$EI = \text{constant}$

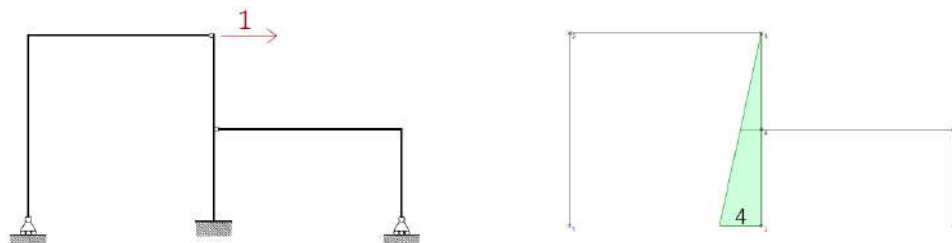
$E = 2 \cdot 10^5 \text{ kg} \cdot \text{cm}^2$

$I = 8 \cdot 10^4 \text{ kg} \cdot \text{cm}^2$

Real system:



Virtual system:



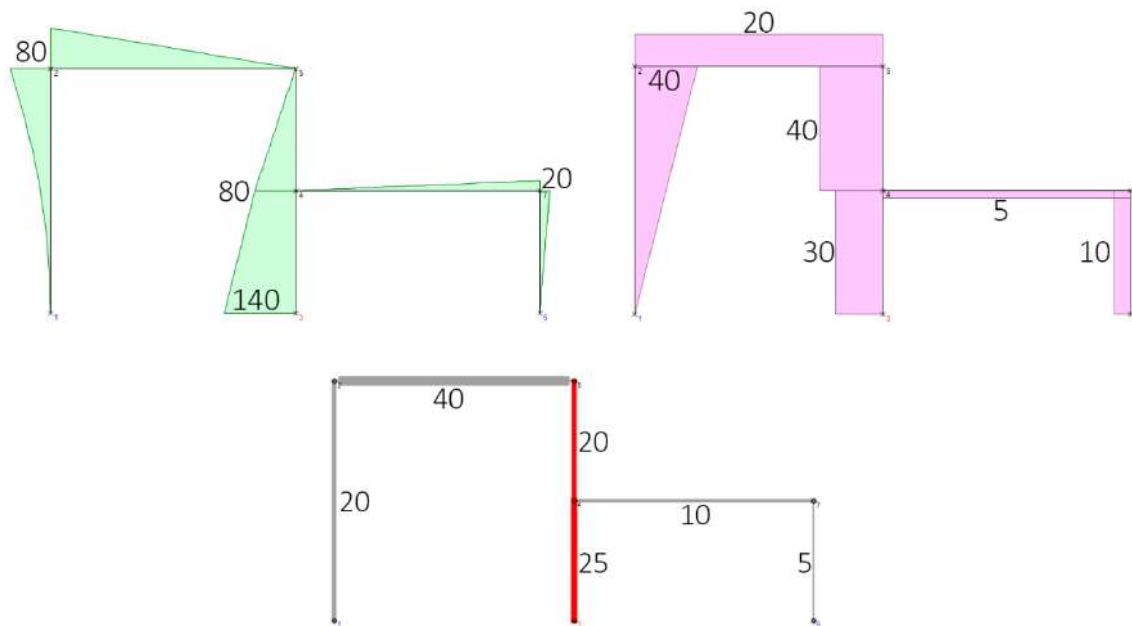
5. joint displacement:

R U	
	$\delta_5 = \left[\left(\frac{1}{3} M \cdot M' \cdot l \right) + \left(\frac{-1}{3} M \cdot M' \cdot l - \frac{1}{6} M \cdot M' \cdot l \right) \right] \cdot \frac{1}{EI}$

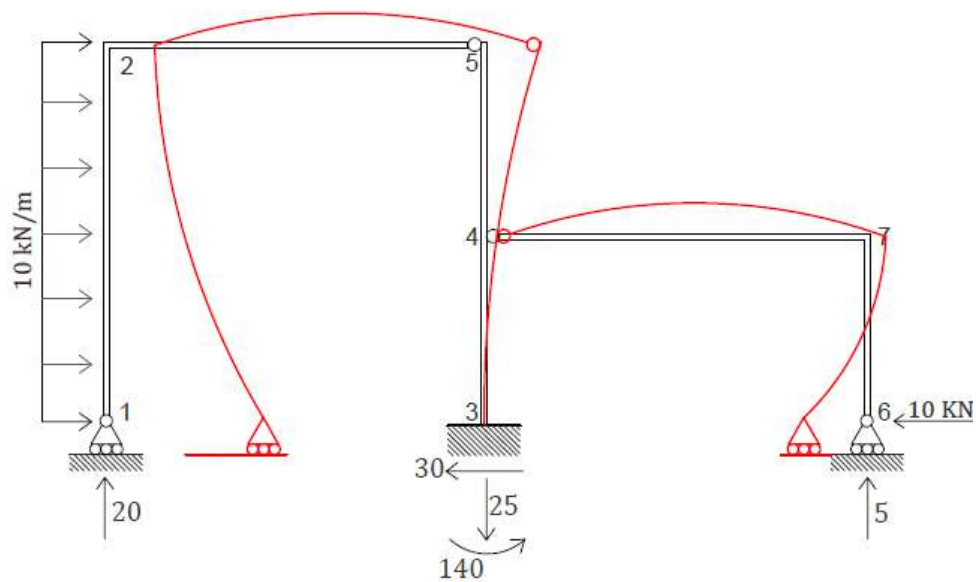
$$\delta_5 = \left[\left(\frac{1}{3} 4 \cdot 160 \cdot 4 \right) + \left(\frac{-1}{3} 4 \cdot 20 \cdot 2 - \frac{1}{6} 2 \cdot 20 \cdot 2 \right) \right] \cdot \frac{1}{EI} = [853, \hat{3} - 53, \hat{3} - 13, \hat{3}] \cdot \frac{1}{EI}$$

$$\delta_5 = \frac{786, \hat{6}}{EI} = 0,02 \text{ m} \rightarrow EI = \frac{786, \hat{6}}{0,02} = 39333, \hat{3} \text{ KN} \cdot \text{m}^2 = 3,9\hat{3} \cdot 10^{13} \text{ N} \cdot \text{mm}^2$$

Diagrams



Reactions and deformation

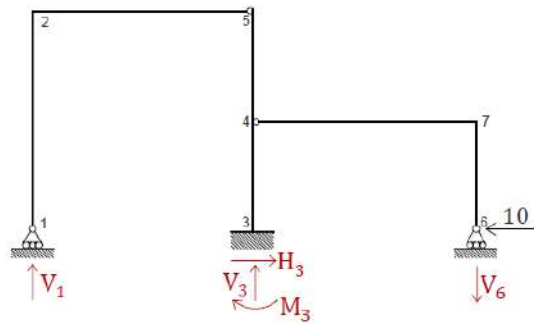


$$\Sigma H = 0 ; 10 \cdot 4 = 30 + 10 ; 40 = 40 \checkmark$$

$$\Sigma V = 0 ; 20 + 5 = 25 ; 25 = 25 \checkmark$$

$$\Sigma M_1 = 0 ; 10 \cdot 4 \cdot 2 + 25 \cdot 4 - 5 \cdot 8 - 140 = 0 ; 80 + 100 - 40 - 140 = 0 ; 0 = 0 \checkmark$$

Resolution of the structure reactions



$$\Sigma H = 0 ; \boxed{H_3 = 10KN}$$

$$\Sigma V = 0 ; V_1 + V_3 - V_6 = 0 ; \boxed{V_3 = -5KN}$$

$$\Sigma M_1 = 0 ; 8 \cdot V_6 + M_3 - 4 \cdot V_3 = 0$$

$$8 \cdot (-5) + M_3 - 4 \cdot (-5) = 0 ; \boxed{M_3 = 20KNm}$$

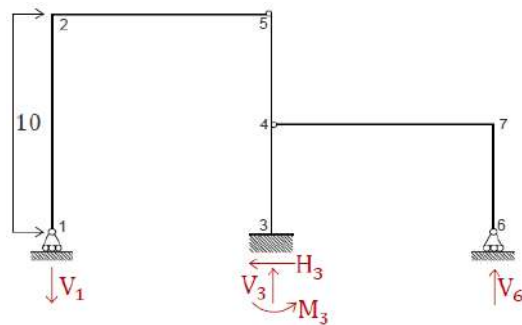


$$\Sigma M_5 = 0 ; \boxed{V_1 = 0KN}$$



$$\Sigma M_4 = 0 ; 10 \cdot 2 + 4 \cdot V_6 = 0$$

$$\boxed{V_6 = -5KN}$$

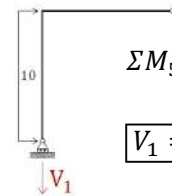


$$\Sigma H = 0 ; \boxed{H_3 = 40KN}$$

$$\Sigma V = 0 ; V_1 = V_3 + V_6 ; \boxed{V_3 = -20KN}$$

$$\Sigma M_1 = 0 ; 40 \cdot 2 - 8 \cdot V_6 - M_3 - 4 \cdot V_3 = 0$$

$$80 - M_3 - 4 \cdot (-20) - 8 \cdot 0 = 0 ; \boxed{M_3 = 160KNm}$$



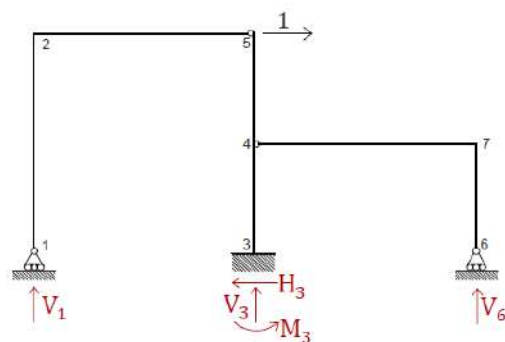
$$\Sigma M_5 = 0 ; 40 \cdot 2 + 4V_1 = 0$$

$$\boxed{V_1 = -20KN}$$



$$\Sigma M_4 = 0 ; 4 \cdot V_6 = 0$$

$$\boxed{V_6 = 0KN}$$

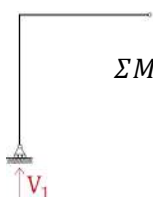


$$\Sigma H = 0 ; \boxed{H_3 = 1KN}$$

$$\Sigma V = 0 ; V_1 + V_3 + V_6 = 0 ; \boxed{V_3 = 0KN}$$

$$\Sigma M_3 = 0 ; 1 \cdot 4 - 4 \cdot V_6 - M_3 + 4 \cdot V_1 = 0$$

$$\boxed{M_3 = 4KNm}$$

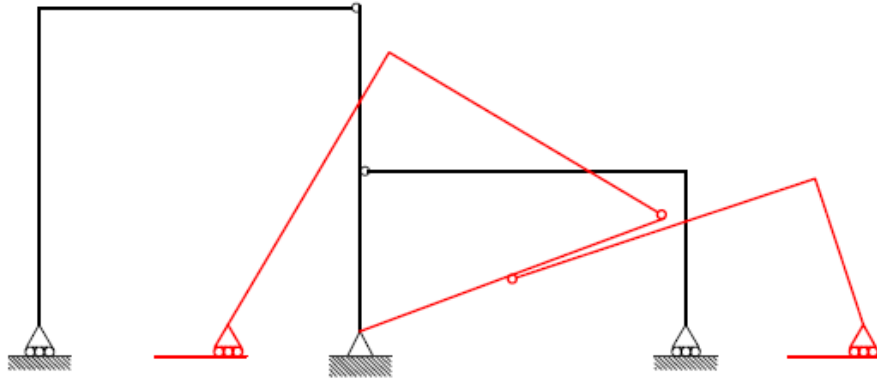


$$\Sigma M_5 = 0 ; \boxed{V_1 = 0KN}$$

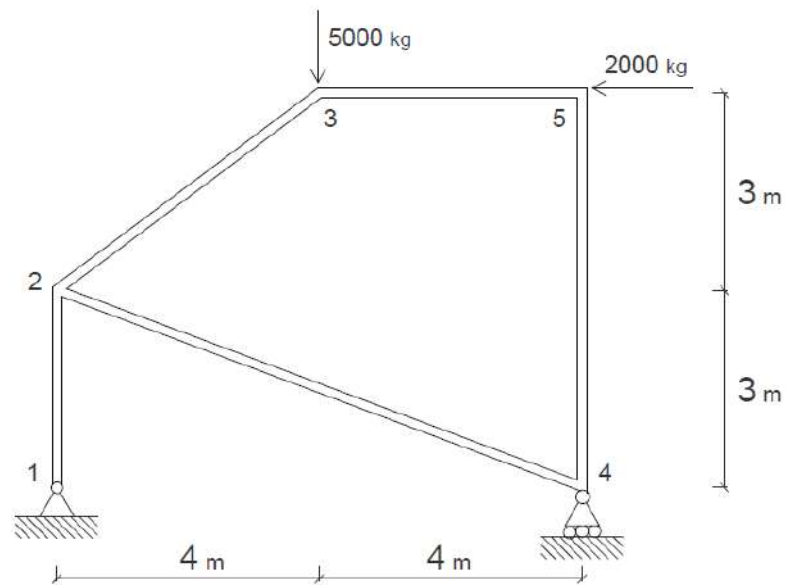


$$\Sigma M_4 = 0 ; \boxed{V_6 = 0KN}$$

If the 3. support was hinged, the frame would become a mechanism. This means that it wouldn't be stable:



6.3. HYPERSTATIC FRAMES. CROSS METHOD



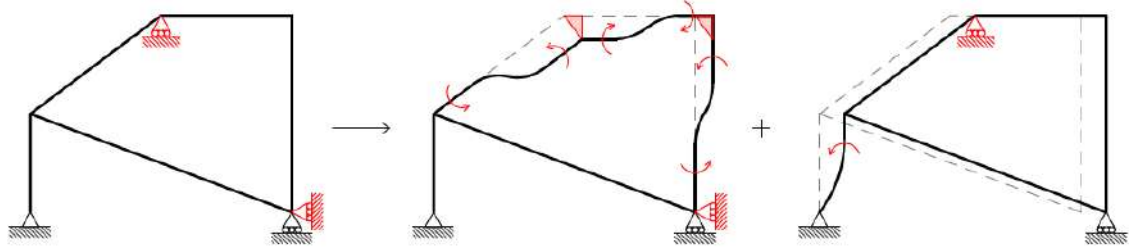
- Using the Cross method, solve the following diagrams: bending moment, shear and axial strengths.
- Solve the reactions and the deformation of the structure.
- Calculate the displacement of the 5 node.

Data:

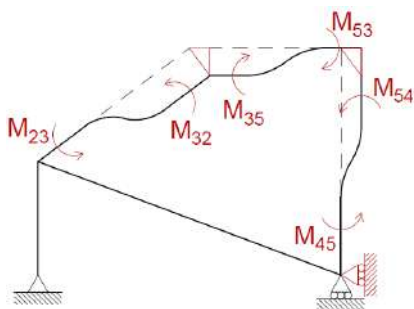
$$EI = \text{constant} = 4300000 \text{ kg} \cdot \text{m}^2$$

Translational degree

$$d = 2n - b - r = 2 \cdot 5 - 5 - 3 = 2$$



1. displacement FEM



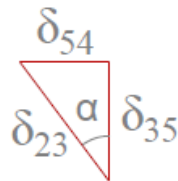
$$\sin \alpha = \frac{3}{5} = 0,6 ; \cos \alpha = \frac{4}{5} = 0,8$$

$$\delta_{23} = \Delta_1 ; \delta_{54} = \delta_{23} \cdot \sin \alpha = 0,6 \Delta_1 ; \delta_{35} = \delta_{23} \cdot \cos \alpha = 0,8 \Delta_1$$

$$M_{23} = M_{32} = \delta_{23} \frac{6EI}{l_{23}^2} = \Delta_1 \cdot \frac{6EI}{5^2} = 0,24 \Delta_1 \cdot EI = 800 \alpha_1$$

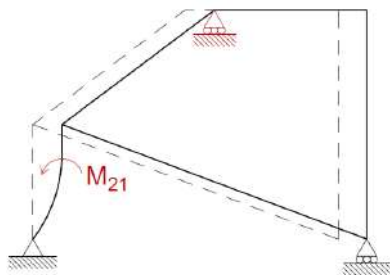
$$M_{54} = M_{45} = \delta_{54} \frac{6EI}{l_{45}^2} = 0,6 \Delta_1 \cdot \frac{6EI}{6^2} = 0,1 \Delta_1 \cdot EI = 333,3 \alpha_1$$

$$M_{35} = M_{53} = -\delta_{35} \frac{6EI}{l_{35}^2} = -0,8 \Delta_1 \cdot \frac{6EI}{4^2} = -0,3 \Delta_1 \cdot EI = -1000 \alpha_1$$



$$\Delta_1 = \frac{3333,3 \alpha_1}{EI}$$

2. displacement FEM



$$M_{21} = \delta_{21} \frac{3EI}{l_{21}^2} = 1000 \alpha_2$$

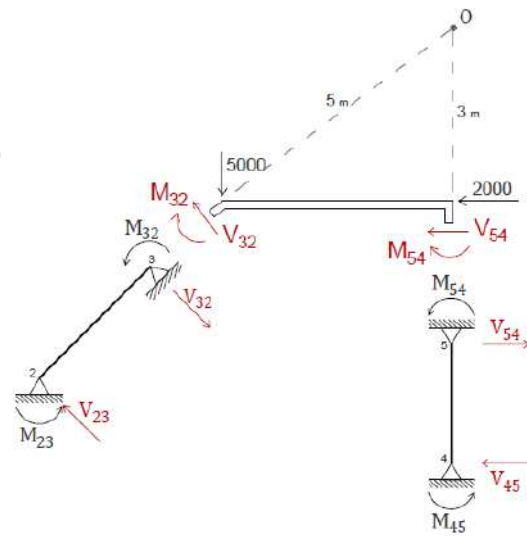
1. equilibrium equation

$$\Sigma M_{21} = 0$$

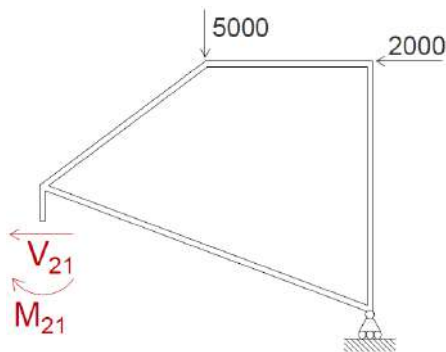
$$M_{32} + M_{54} + 5 V_{32} + 3 V_{54} + 2000 \cdot 3 - 5000 \cdot 4 = 0$$

$$M_{32} + M_{54} + 5 \frac{M_{32} + M_{23}}{5} + 3 \frac{M_{54} + M_{45}}{6} = 14000$$

$$M_{23} + 2 M_{32} + 0,5 M_{45} + 1,5 M_{54} = 14000$$



2. equilibrium equation



$$\Sigma H = 0$$

$$V_{21} + 2000 = 0 ; \frac{M_{21}}{3} + 2000 = 0 ; \boxed{M_{21} = -6000}$$

Stiffness

$$K_{21} = \frac{3EI}{l} = EI ; K_{24} = \frac{4EI}{l} = 0,468 EI ; K_{23} = \frac{4EI}{l} = 0,8 EI$$

$$K_{35} = \frac{4EI}{l} = EI ; K_{54} = \frac{4EI}{l} = 0,6 EI$$

Distribution factors

-2. node:

$$r_{21} = \frac{K_{21}}{K_{21} + K_{23} + K_{24}} = \frac{EI}{(1 + 0,468 + 0,8)EI} = 0,44$$

$$r_{23} = \frac{K_{23}}{K_{21} + K_{23} + K_{24}} = \frac{0,8 EI}{(1 + 0,468 + 0,8)EI} = 0,35 ; r_{24} = 1 - r_{21} - r_{23} = 0,21$$

-3. node:

$$r_{32} = \frac{K_{32}}{K_{32} + K_{35}} = \frac{0,8 EI}{(1 + 0,8)EI} = 0,46; r_{35} = 1 - r_{32} = 0,54$$

-5. node:

$$r_{53} = \frac{K_{53}}{K_{53} + K_{54}} = \frac{EI}{(1 + 0,6)EI} = 0,62; r_{54} = 1 - r_{53} = 0,38$$

-4. node:

$$r_{42} = \frac{K_{42}}{K_{42} + K_{45}} = \frac{0,468 EI}{(0,468 + 0,6)EI} = 0,44; r_{45} = 1 - r_{42} = 0,56$$

Equation resolution

$$M_{23} + 2 M_{32} + 0,5 M_{45} + 1,5 M_{54} = 14000; 2851,55 \alpha_1 - 492,7 \alpha_2 = 14000$$

$$M_{21} = -6000; -333,3 \alpha_1 + 527,7 \alpha_2 = -6000$$

$$\rightarrow \boxed{\alpha_1 = 3,237; \alpha_2 = -9,325}$$

$$M_{23} = 571 \alpha_1 - 332,9 \alpha_2$$

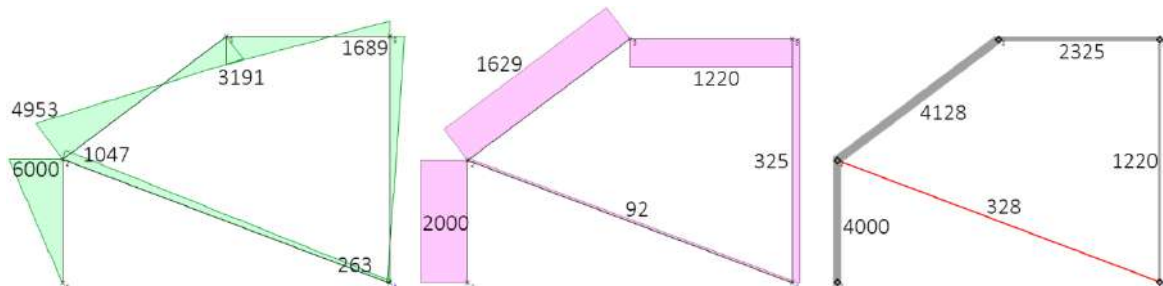
$$M_{32} = 717,9 \alpha_1 - 93 \alpha_2$$

$$M_{45} = 245,3 \alpha_1 + 56,9 \alpha_2$$

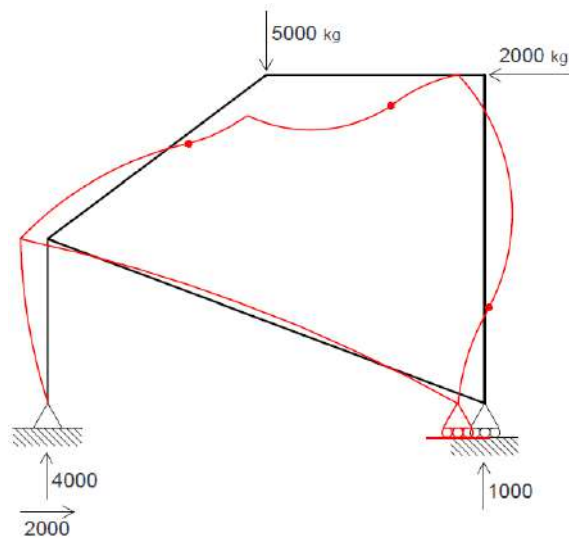
$$M_{54} = 517,4 \alpha_1 - 1,5 \alpha_2$$

$$M_{21} = -333,3 \alpha_1 + 527,7 \alpha_2$$

Diagrams



Reactions and deformation



5. joint displacement

$$\delta_5 = \delta_{54} (1. \text{ disp.}) + \delta_{21} (2. \text{ disp.})$$

$$\delta_5 = \frac{M_{54} \cdot l_{54}^2}{6EI} + \frac{M_{21} \cdot l_{21}^2}{3EI} = \frac{333,3 \alpha_1 \cdot 6^2}{6EI} + \frac{1000 \alpha_2 \cdot 3^2}{3EI} = \frac{6 \cdot 333,3 \cdot 3 \cdot 2,37 + 1000 \cdot 3 \cdot (-9,325)}{EI}$$

$$\delta_5 = \frac{-21321}{EI} = \frac{-21321}{4,3 \cdot 10^6} = 5 \cdot 10^{-3} \text{ m} = 0,5 \text{ cm (to the left)}$$

1.STATE (1.DISP)											
1		2			3			5		4	
12		21	24	23	32	35		53	54	42	45
-		0,44	0,21	0,35	0,44	0,56		0,60	0,40	0,41	0,59
FEM x a1	800,0			800,0	800,0	-1000,0		-1000,0	333,3		333,3
	-800,0	-352,7	-165,1	-282,2	-341,1	-141,1				-82,6	
				75,8	341,1	151,6	189,5	-571,9	94,7		
							171,6	571,9	343,2	228,8	365,2
	0,5		-75,3							-107,3	-365,2
	-0,5	-0,2	-0,1	-0,2	171,5	-0,1					-150,6
				-38,1	-171,5	-76,2	-95,3	-154,9	-47,6		0,0
							46,5	154,9	92,9	62,0	30,9
	-44,5		-6,4							-9,1	-30,9
	44,5	19,6	9,2	15,7	54,3	7,8					-12,8
					-54,3	-24,1	-30,2	-24,2	-15,1		4,6
								24,2	14,5	9,7	9,4
										-9,4	-3,9
											-5,5
	0,0		-333,3	-237,7	571,0	717,9	-717,9	-517,4	517,4	-245,3	245,3
2.STATE (2.DISP)											
1		2			3			5		4	
12		21	24	23	32	35		53	54	42	45
-		0,44	0,21	0,35	0,44	0,56		0,60	0,40	0,41	0,59
FEM x a2	1000,0	1000,0									
	-1000,0	-440,9	-206,4	-352,7	-176,4	-176,4				-103,2	
				39,2	176,4	78,4	98,0	49,0	49,0		
							-14,7	-49,0	-29,4	-19,6	-113,0
	62,5		23,3							33,2	113,0
	-62,5	-27,6	-12,9	-22,0	-25,7	-11,0					46,6
				5,7	25,7	11,4	14,3	40,3	7,1		-6,5
							-12,1	-40,3	-24,2	-16,1	-14,5
	8,7		3,0							4,3	14,5
	-8,7	-3,8	-1,8	-3,1	-13,6	-1,5					6,0
					13,6	6,1	7,6	8,1	3,8		-0,9
								-8,1	-4,8	-3,2	-2,5
										2,5	1,0
											1,5
	0,0		527,7	-194,8	-332,9	-93,0	93,0	1,5	-1,5	-56,9	56,9
3.STATE x a1 a1=3,237											
1		2			3			5		4	
12		21	24	23	32	35		53	54	42	45
0		-1079	-769	1848	2324	-2324		-1675	1675	-794	794
3.STATE x a2 a2=-9,325											
1		2			3			5		4	
12		21	24	23	32	35		53	54	42	45
0		-4921	1817	3105	868	-868		-14	14	531	-531
FINAL RESULTS											
1		2			3			5		4	
12		21	24	23	32	35		53	54	42	45
0		-6000	1047	4953	3191	-3191		-1689	1689	-263	263

7. COMPUTER CALCULATION OF STRUCTURES BY MATRIX METHOD. WINEVA

In the following chapter, we are going to learn using the Wineva program for calculation of frames. We will do that through a tutorial explaining a practical exercise with an easy frame.

But before beginning with the exercise, let's explain what the process to describe nodes and bars in this software is.

7.1. NODE'S DEFINITION

When introducing nodes (joints or supports) in the Wineva program, the main feature we have to focus on is the freedom degree. In other words, we have to describe if nodes can rotate or translate in any of the two directions of the two-dimensional space.

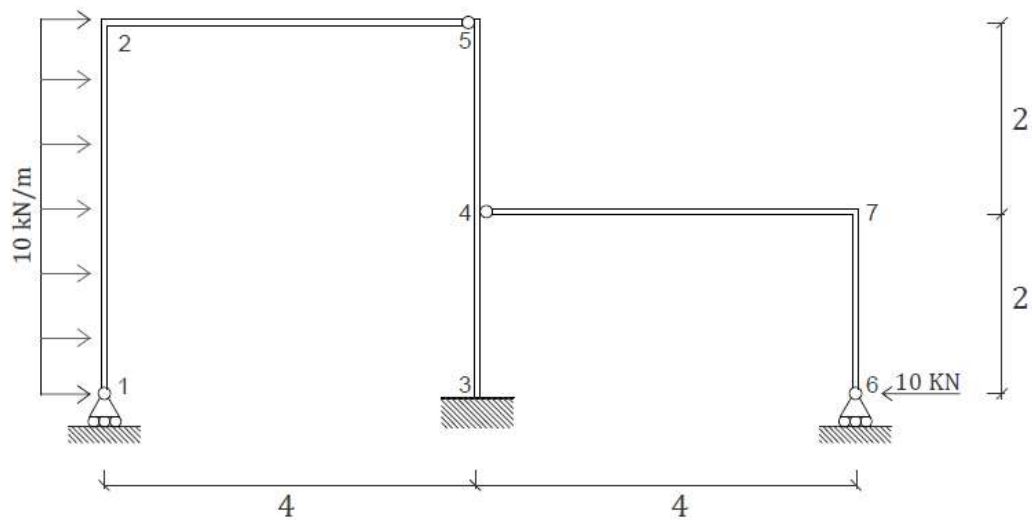
It means that from the point of view of freedom degree, hinged joints and stiff nodes are equal. Both of them have three degrees of freedom. The way to represent the different behaviour of these two types of nodes is through the definition of the bar's ends.

7.2. BAR'S DEFINITION

When describing a frame in the Wineva program, the way to differentiate hinged joints from stiff joints is through the bar's features. When creating a bar, the program describes boundary conditions for the ends of each member. By default, it creates bars with stiff ends (ends with the ability to transmit moments).

By changing these features and defining pin ends in the bars, we can create hinged joints.

7.3. EXERCISE



Using Wineva program, get the reaction and deformation of the structure.

Obtain the bending moment, shear and axial strength diagrams.

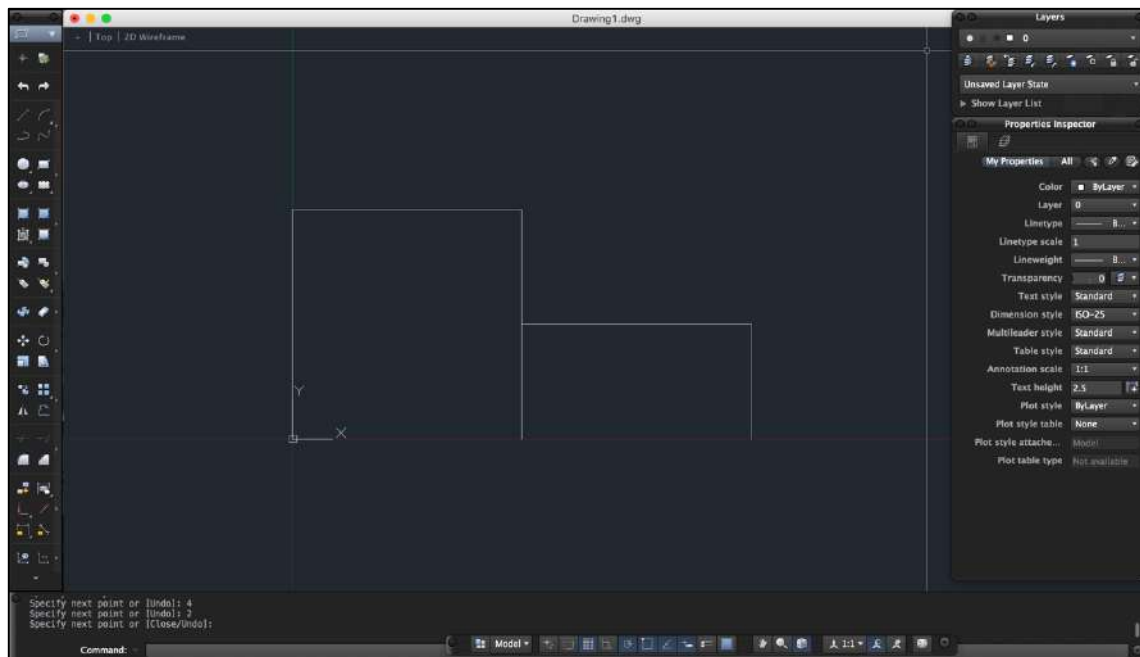
Data:

$$EI = \text{constant}$$

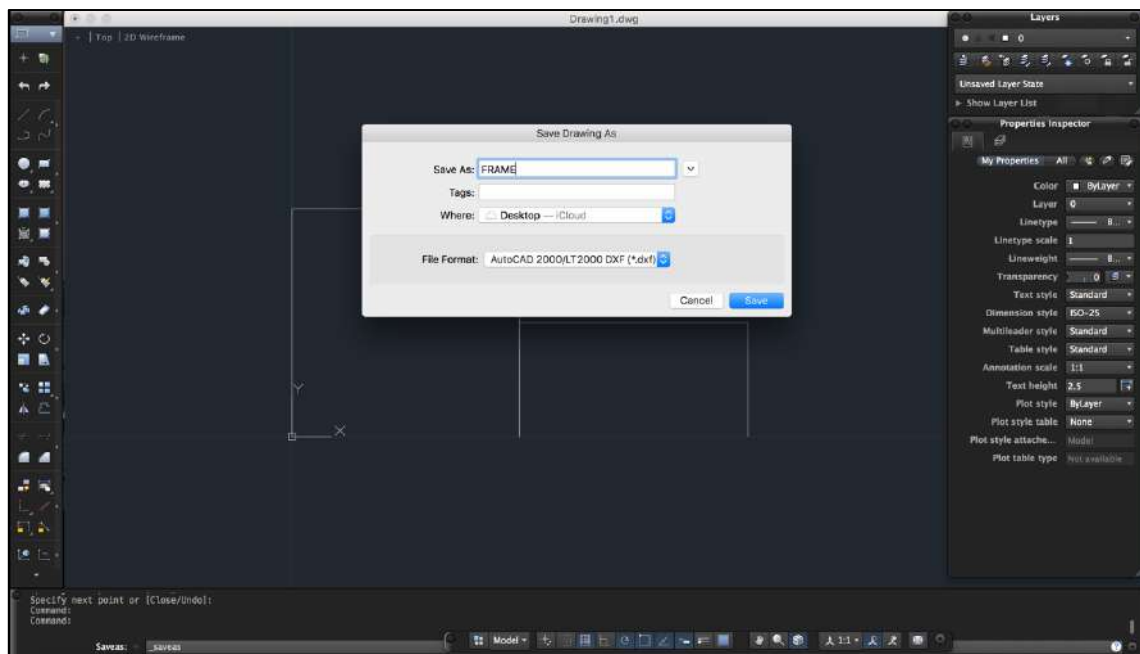
$$E = \text{Steel} = 2 \cdot 10^6 \text{ kg} \cdot \text{cm}^2$$

$$I = \text{HEB300}$$

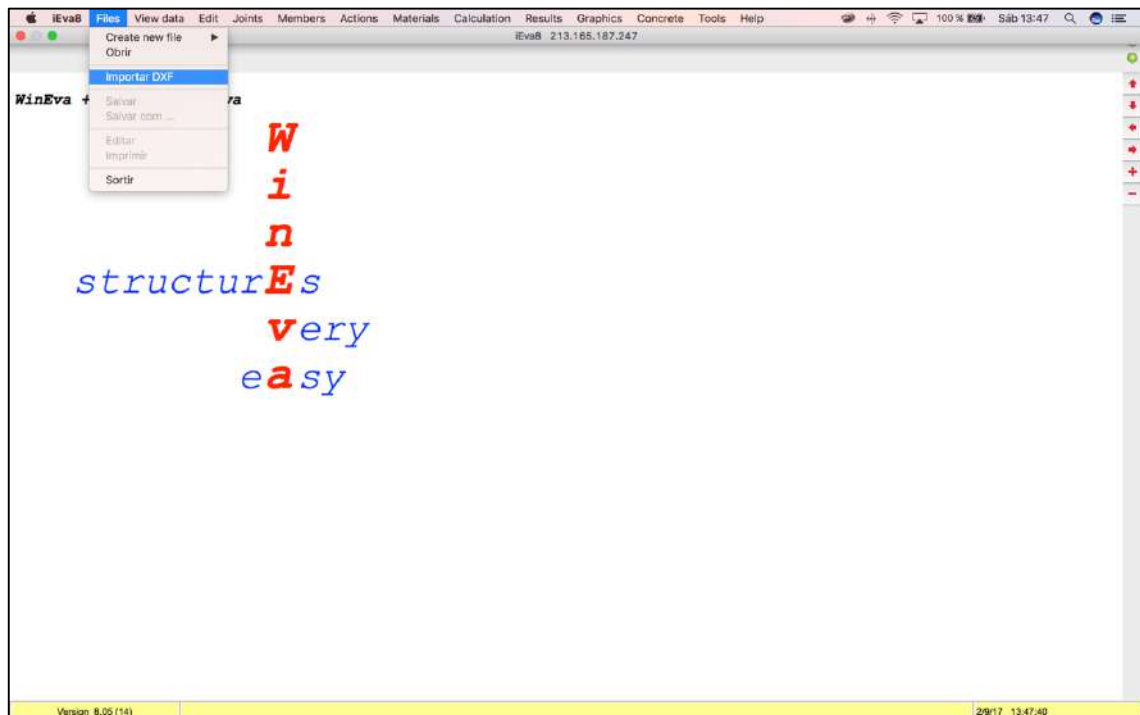
We first draw the geometry of the frame we want to calculate in Autocad. It is better if we locate one of the corners of our frame in the origin of the coordinate system of the Autocad's model space. When drawing the frame, we need to focus on making an independent line with every bar length between nodes.



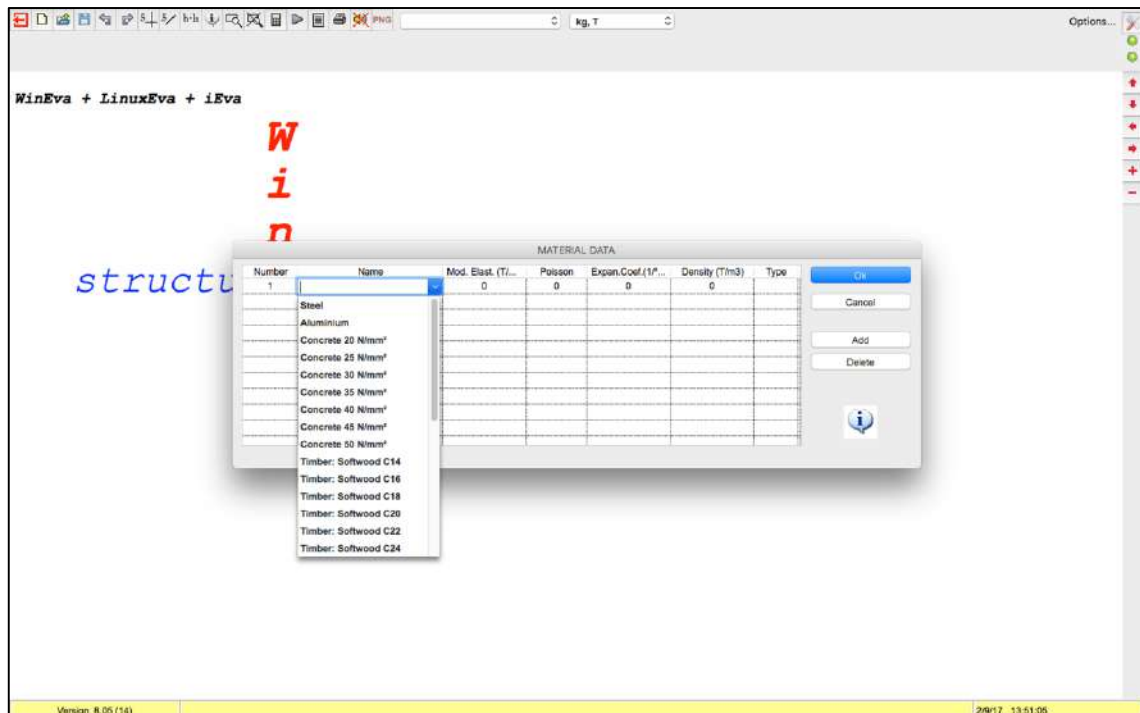
Once the frame is drawn, save it like a “.dxf” file



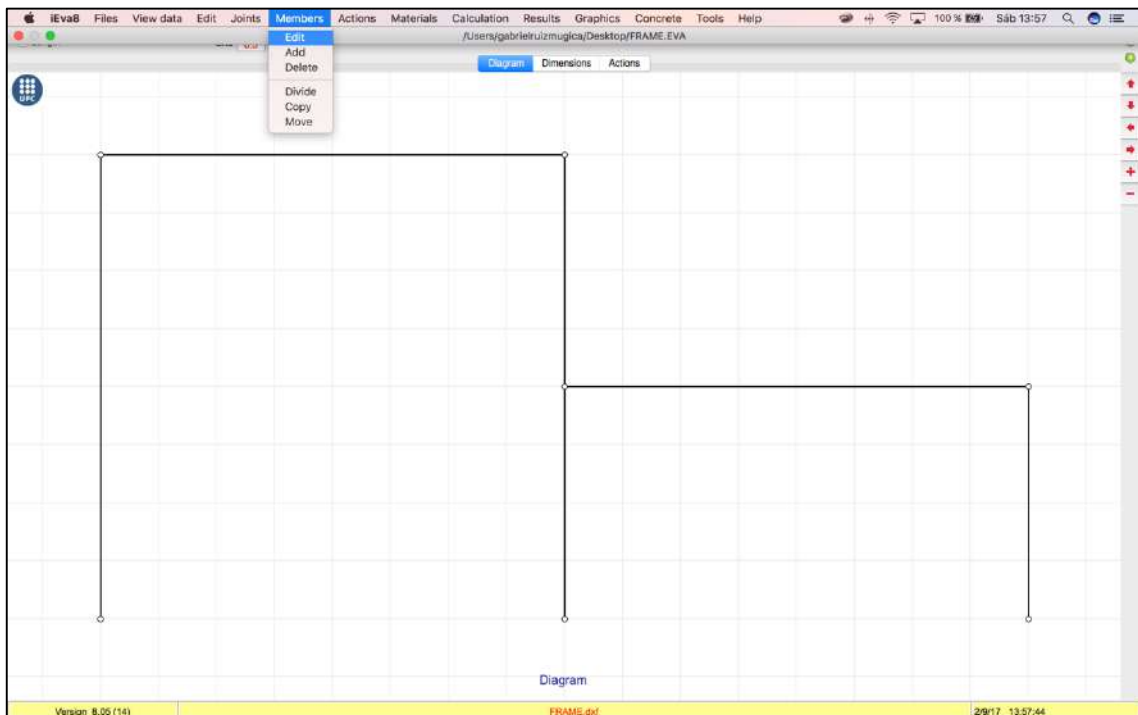
Now let's open Wineva program. Open the "file" menu and click on "Import dxf". Then open the frame we have created.



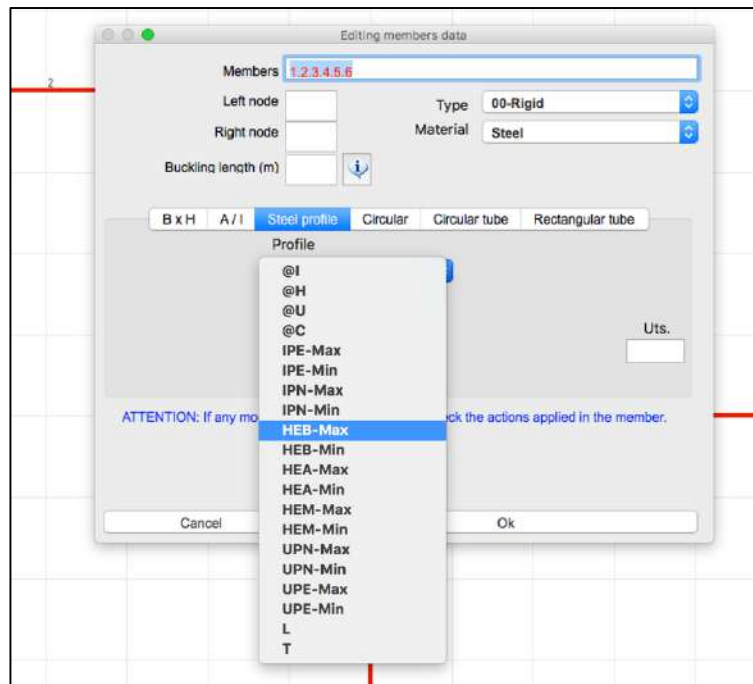
When opening the frame, the program asks about the material the structure is made of. In this case, we have to choose "Steel".



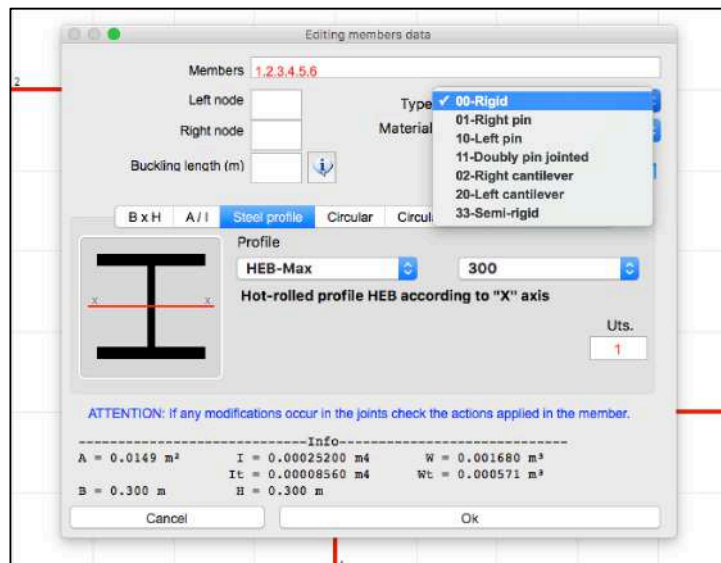
After opening the frame we have to first describe the bar's features. We can do it by clicking on "members" menu, and going to "edit".



All the bars in our structure are HEB300 profile type, so we select all of them, and by opening the "select profile" menu, we click on "HEB-Max".

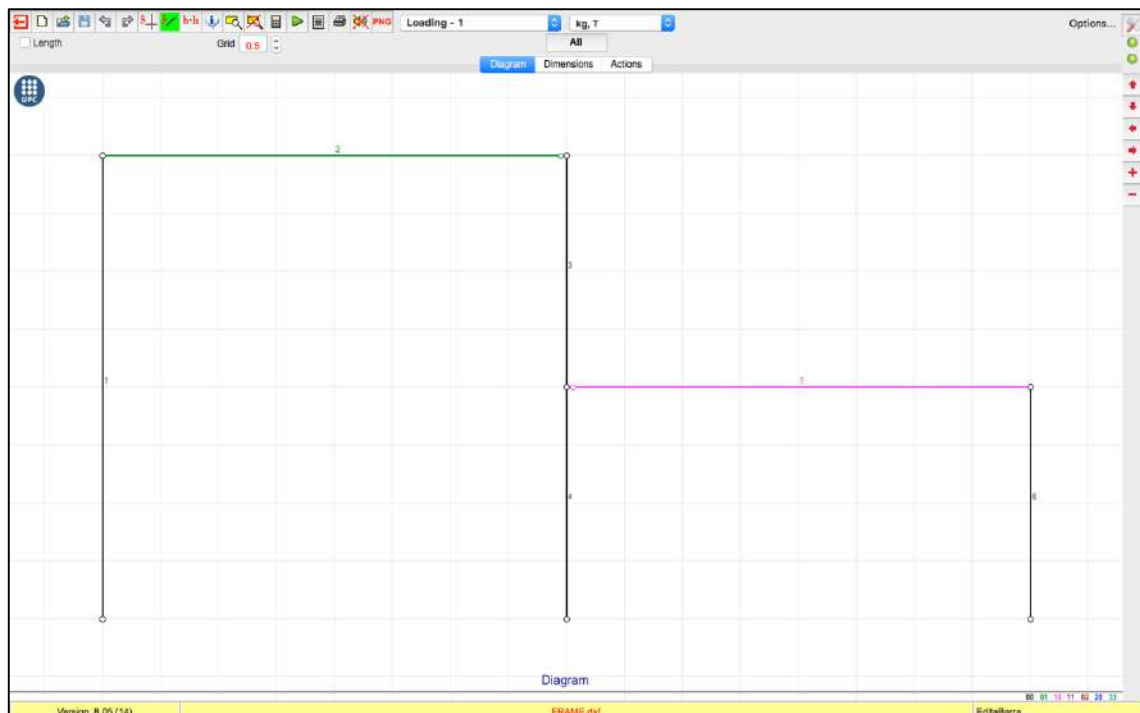


We should note that when editing members of a frame, the program automatically creates bars with stiff ends.

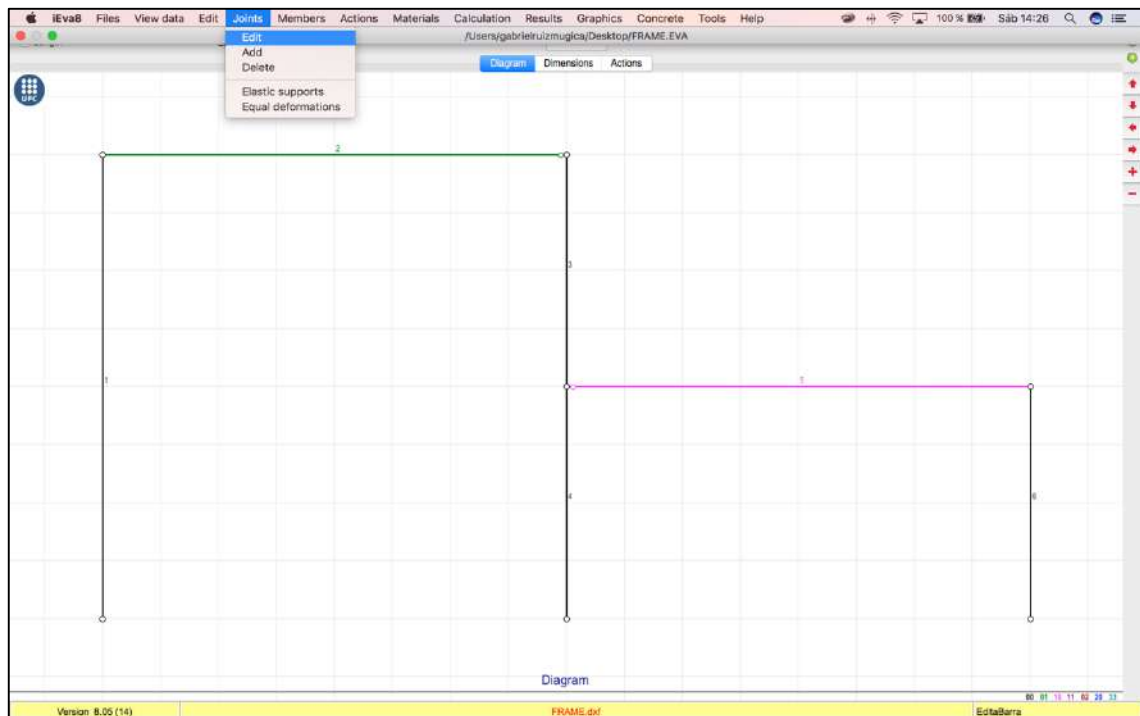


But in the frame we are describing 2-5 bar's right end and 4-7 bar's left end are hinges (pins). So we need to edit these two bars and click on "Right pin" for 2-5 bar, and "Left pin" for 4-7 bar.

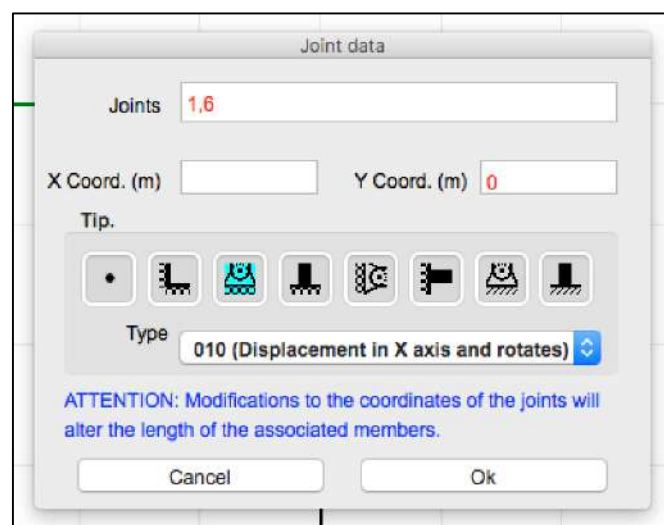
Now the frame looks like that



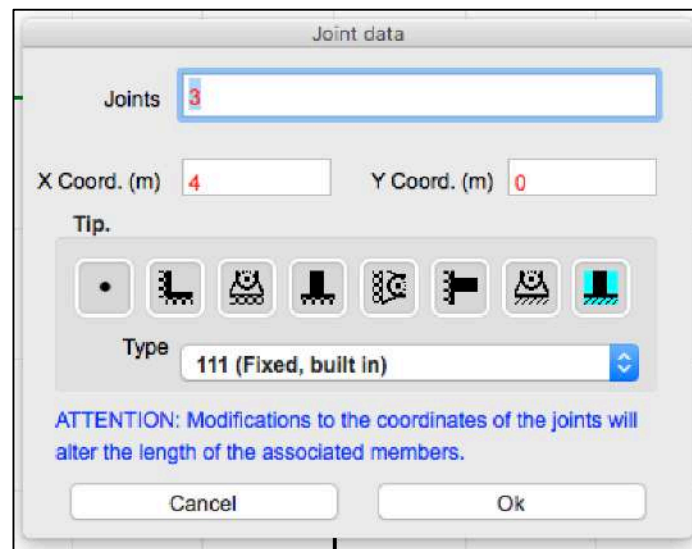
The next step is to describe nodes features. The program automatically creates free nodes (three degrees of freedom). So, by default, all the joints of our frame are right. We only need to define the supports. In order to do that, let's open "Joints" menu and click on "Edit".



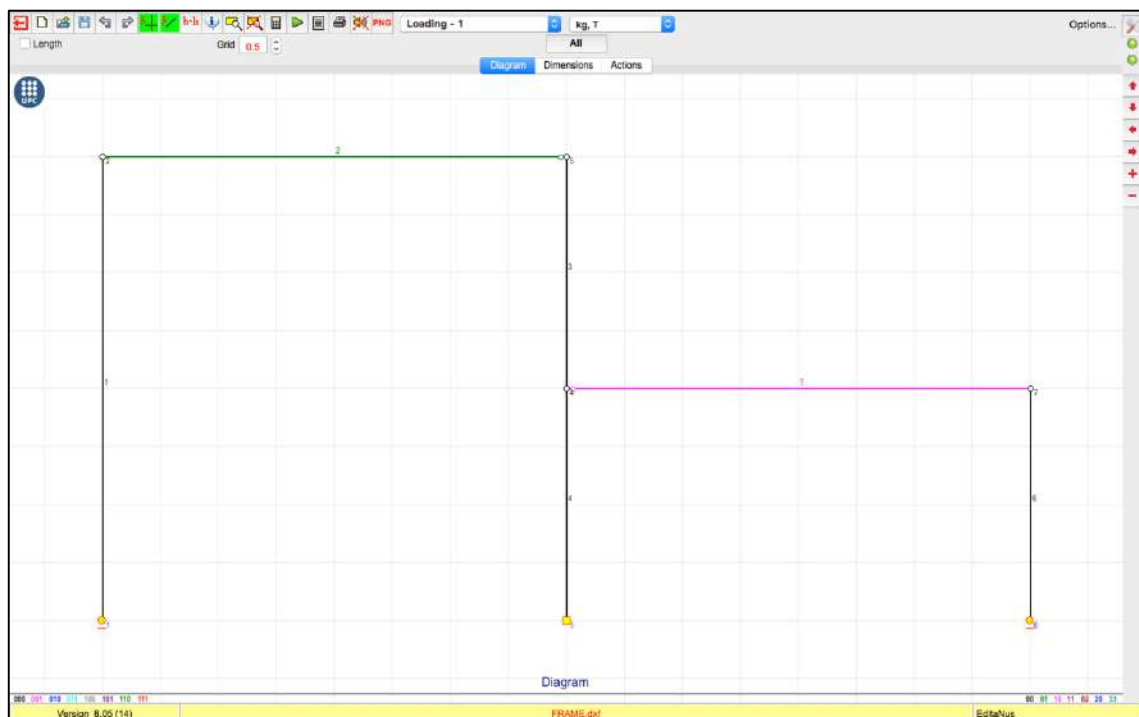
Then we must select the node we want to define, and click on the type of support that matches with the degree of freedom we are looking for. Let's start with 1 and 6 nodes (roller supports).



Now we will define the node 3 (Fixed end support).



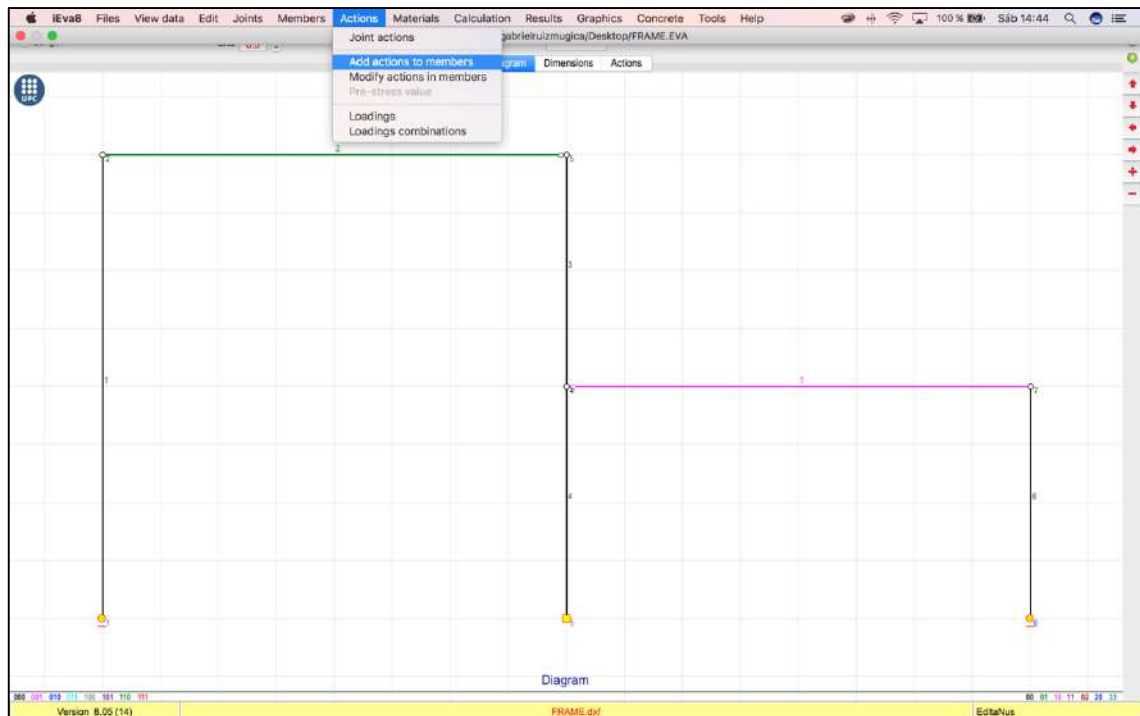
And Now the frame looks like that.



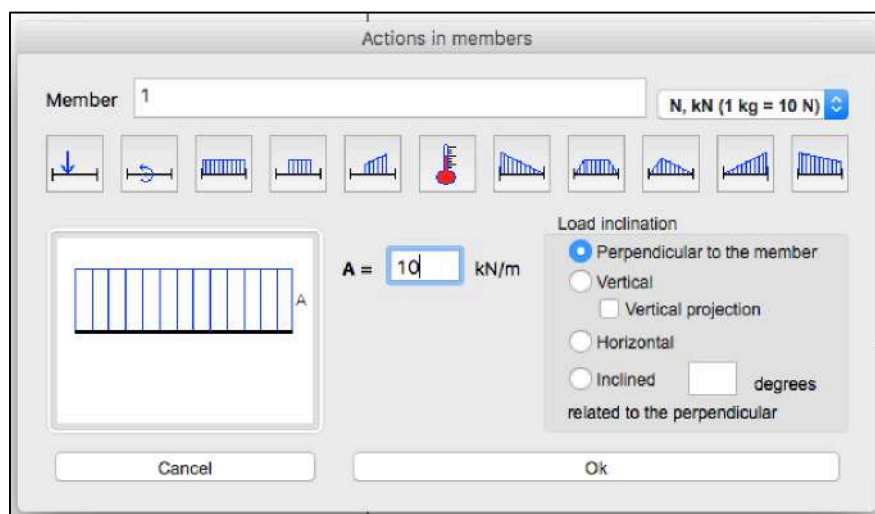
Frame's geometry and physical properties are already defined. The next step is to introduce the loads.

Our frame has two kinds of loads: a distributed load on 1-2 bar, and a point load in node 6.

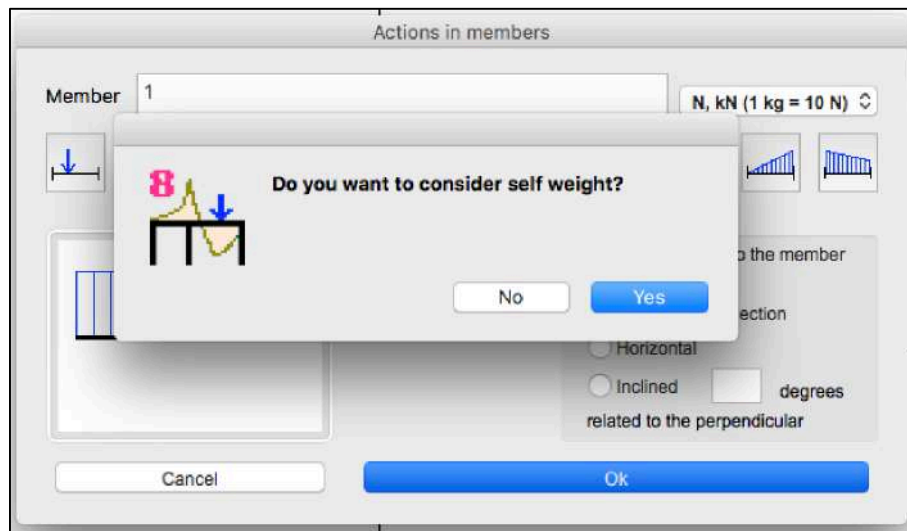
Let's start with 1-2 bar. To introduce a distributed load we must open the "Actions" menu and click on "Add actions to members"



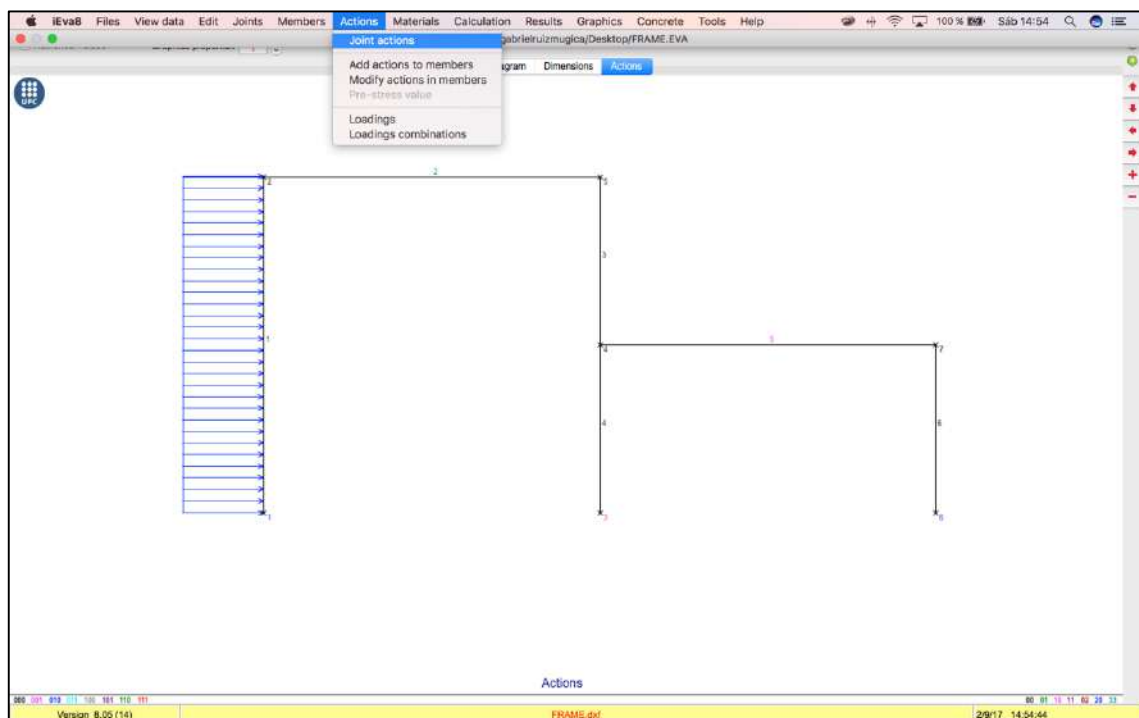
Once the bar is selected, we have to introduce the units in the window (kN/m in this case) and the value: 10 kN/m.



When introducing the first load, the program asks if we want to take into account the self weight of the frame. If we are calculating a frame by computer to compare with hand made calculation, it is better to answer no, because otherwise, it is going to create a deviation between these two calculations.



Let's continue with node 6. To introduce a point load we must open the "Actions" menu and click on "Joint actions"



Once the node is selected, we have to introduce the units and the value in the window: 10 kN. By default, positive values indicate that load direction is towards the right, so we will introduce -10kN, meaning that our load acts towards the left.

Actions in joints

JOINT: 6

N, kN (1 kg = 10 N)

Forces and applied moments

Fx: -10 kN

Fy: kN

Mz: kNm

Forced displacement

Dx: m

Dy: m

Gz: Rad

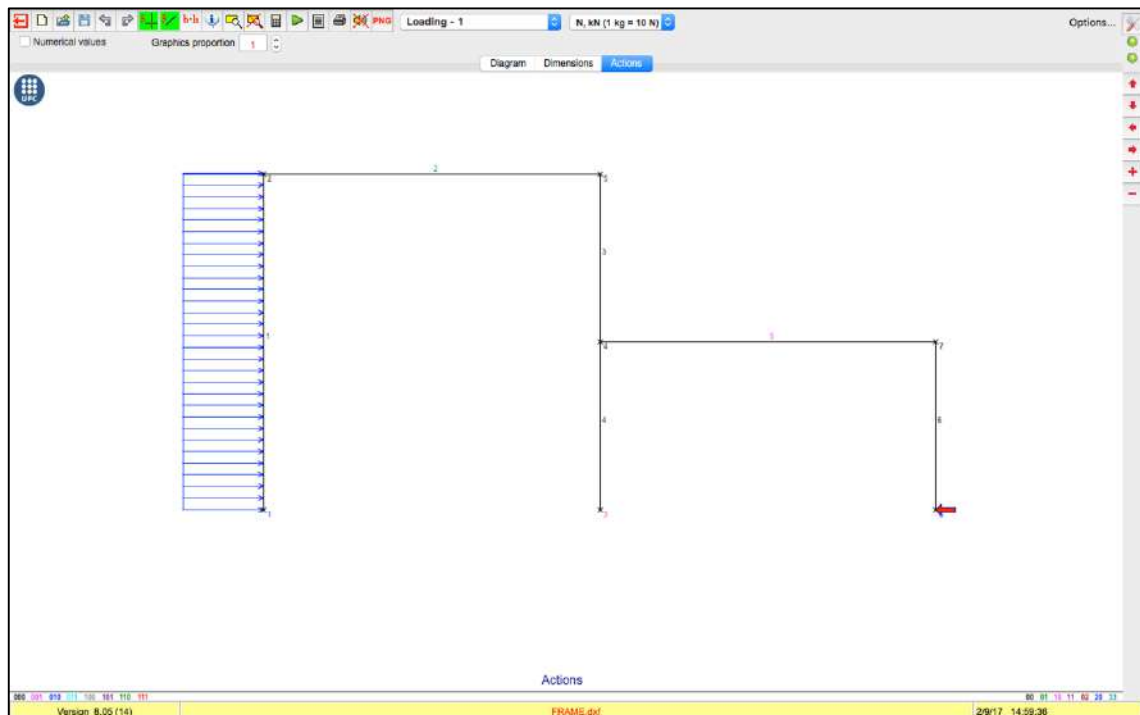
OK (continue)

Cancel (continue)

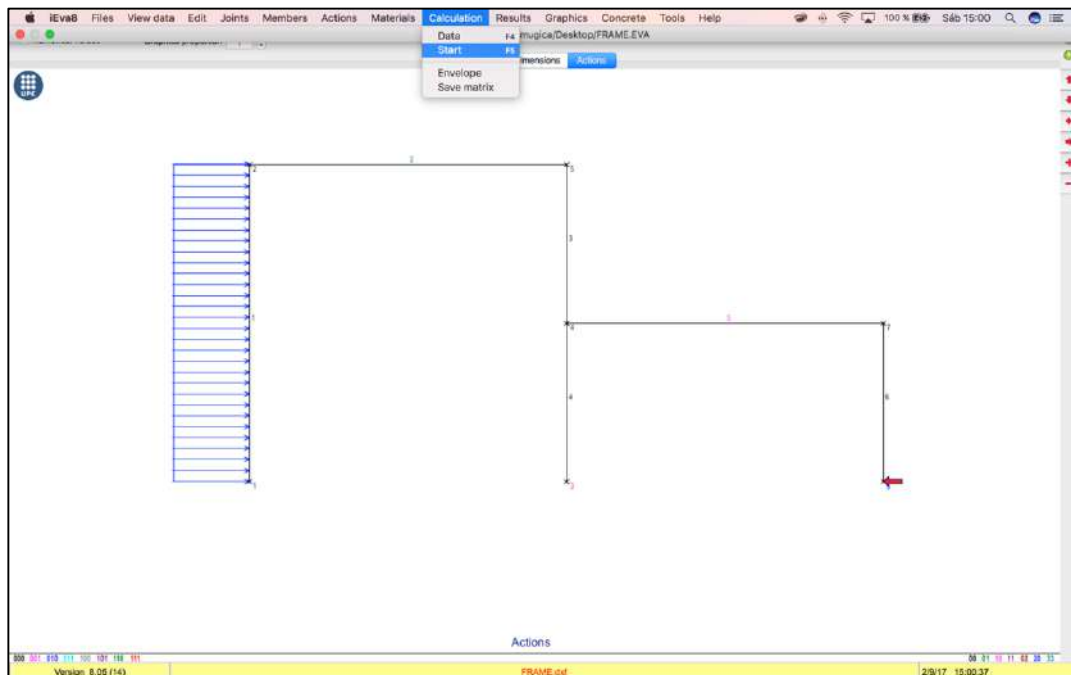
OK (end)

Cancel (end)

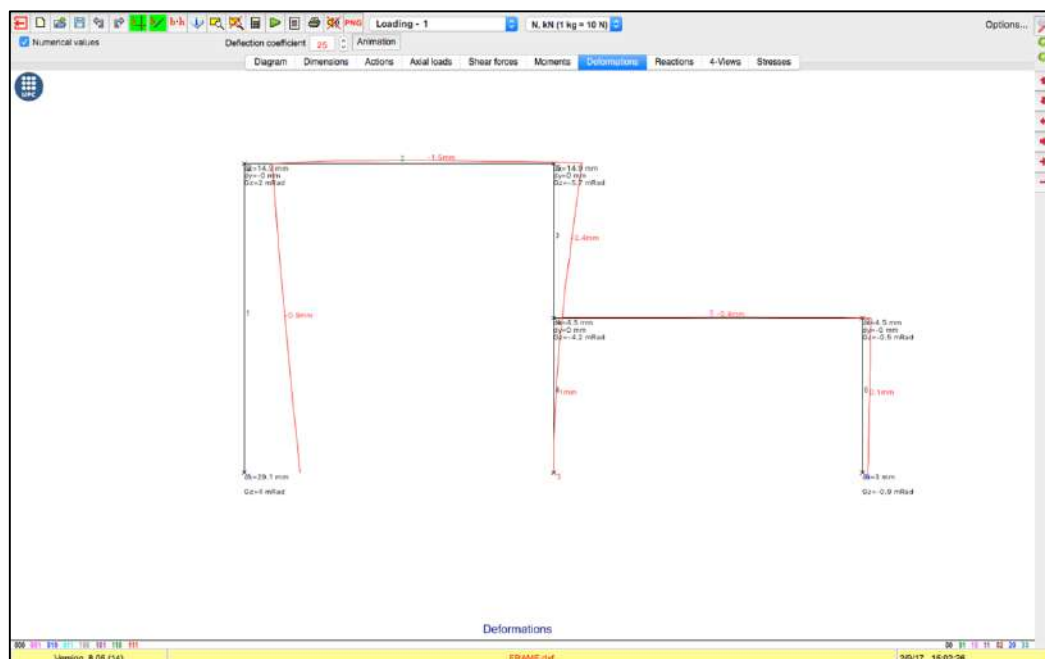
The frame is ready to be calculated. That's how it looks like now.



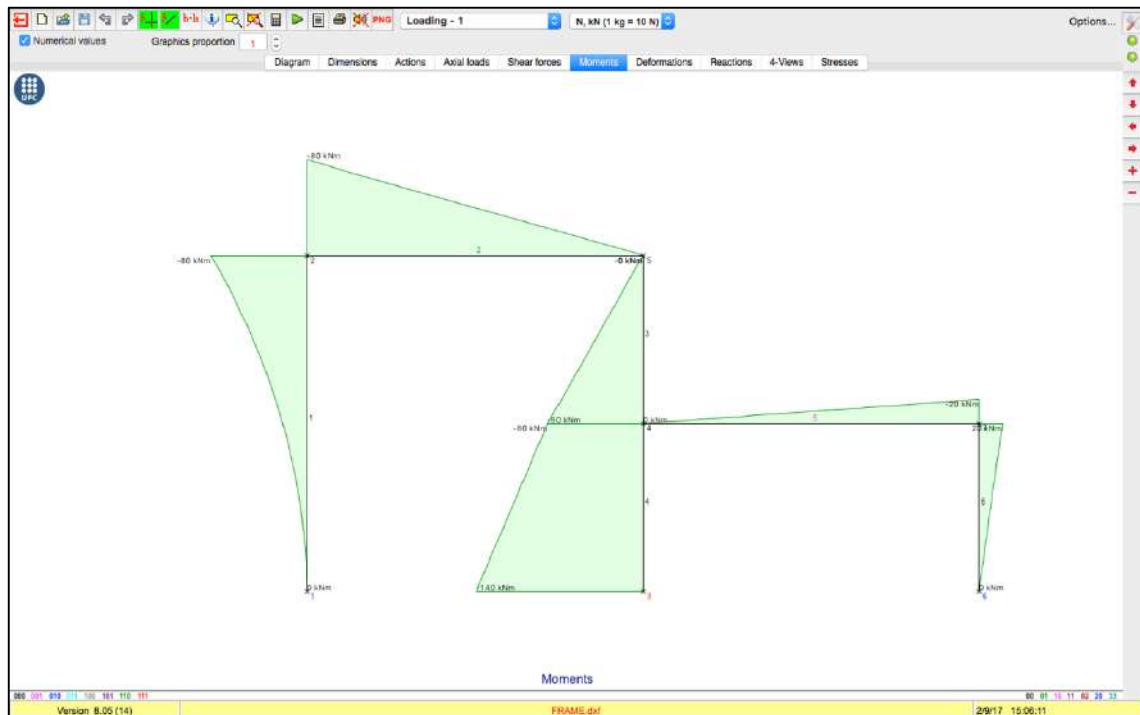
For calculating the frame, we have to open the “Calculation” menu and click on “Start”.



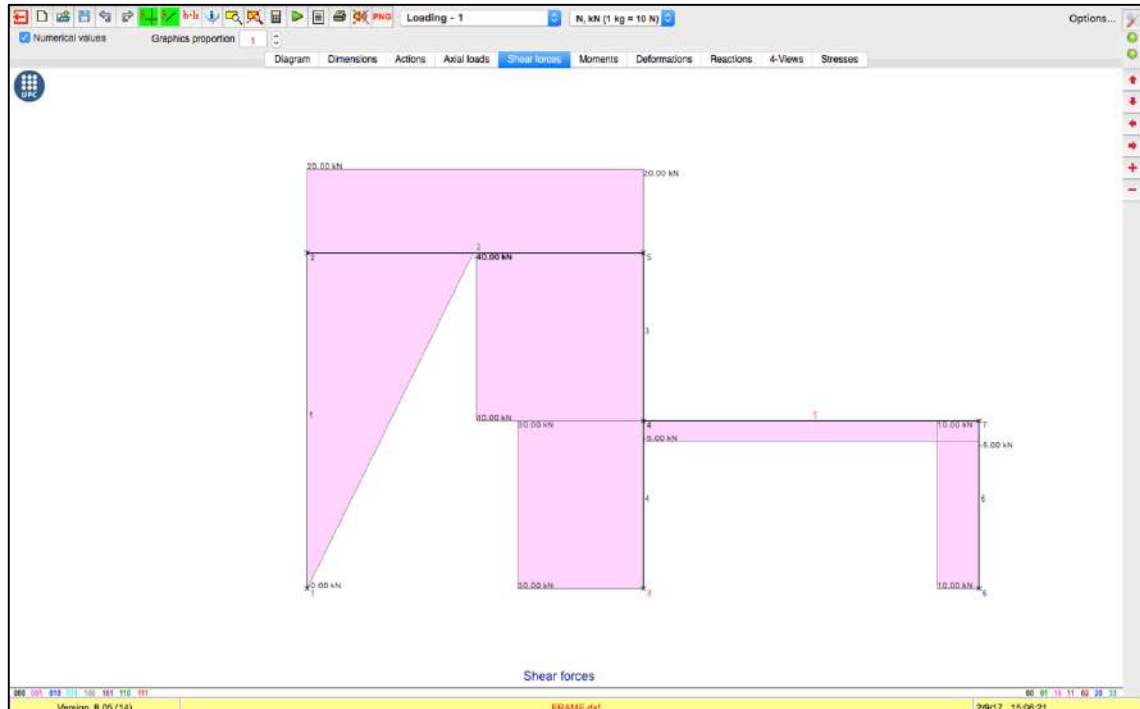
Once calculated, the program offers many windows with different information. It is interesting to activate the “numerical values” tab, in the upper-left corner of the screen, to see the value of all loads, reactions and strengths. The main windows we have are: Deformation.



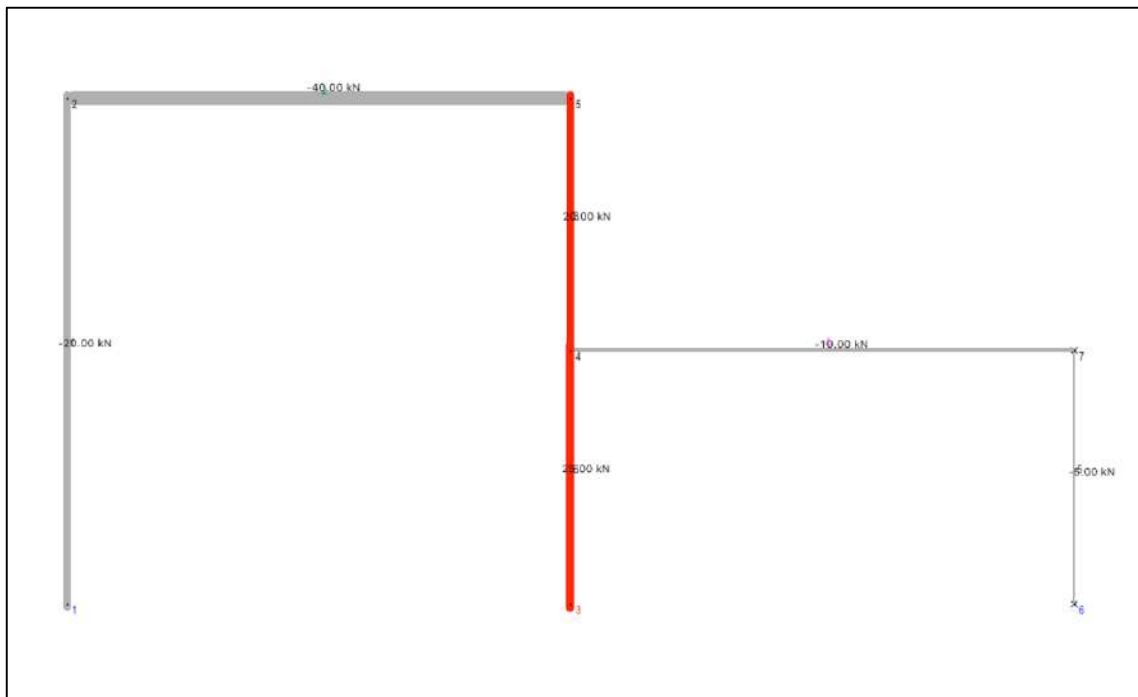
Bending moment strengths



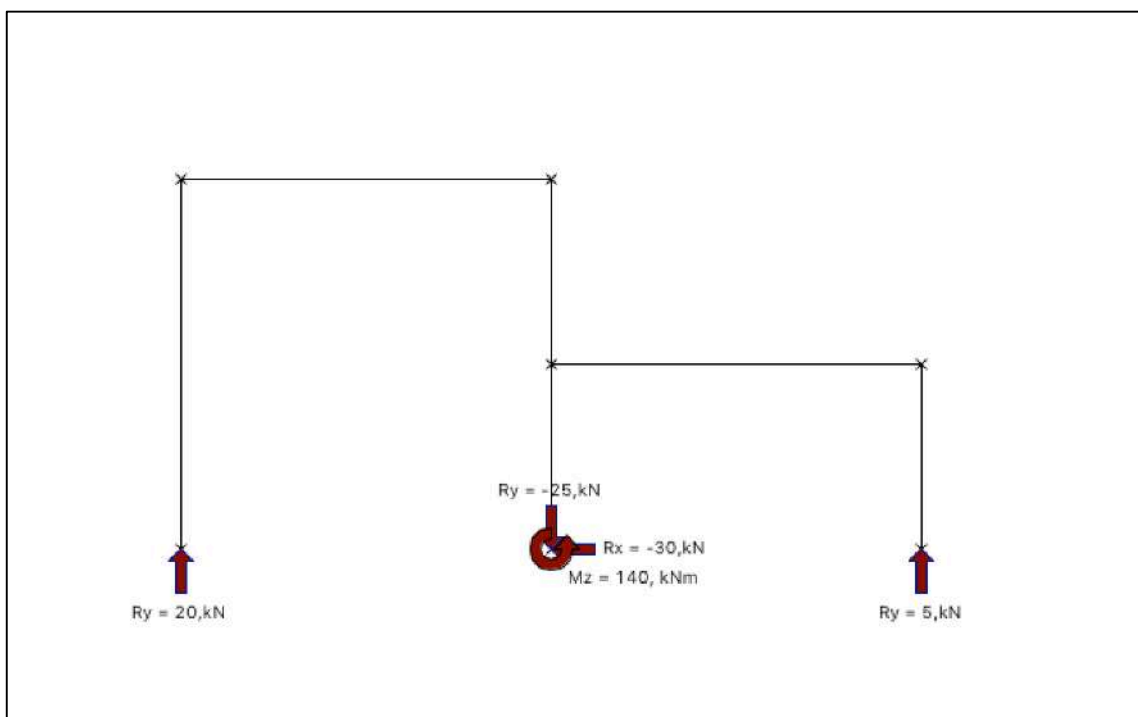
Shear strength



Axial strength



And reactions



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Ed.: Horacio Escarabajal Editores

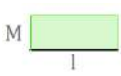
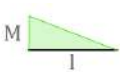
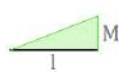
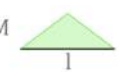
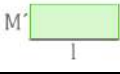
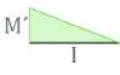
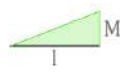
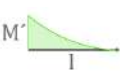
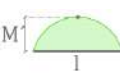
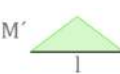
“Strutture. Morfología strutturale in architettura”

Mario Cedolini
Ed.: Arsenale Editoriale

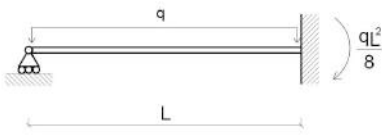
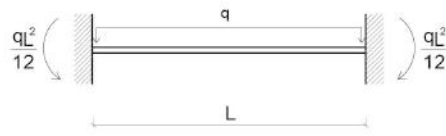
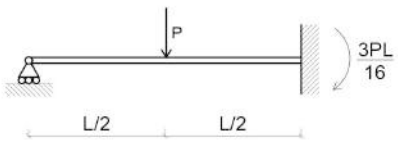
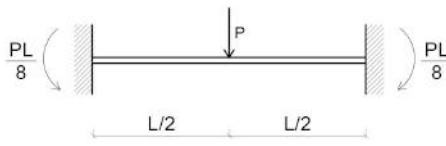
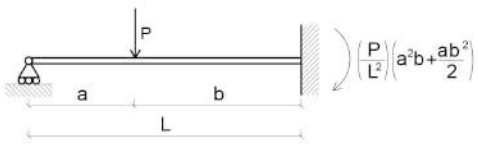
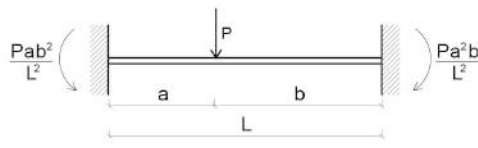
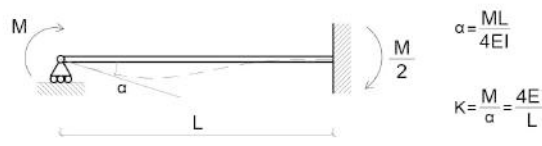
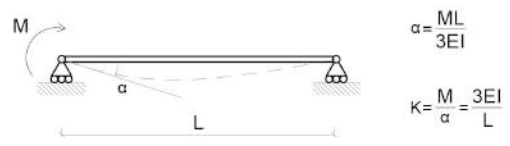
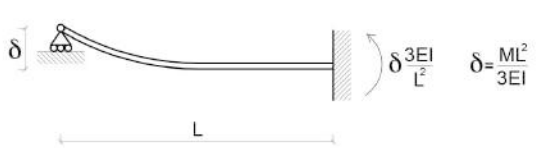
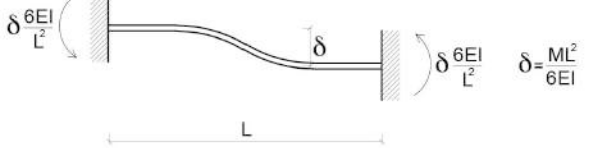
“Números gordos en el proyecto de estructuras”

Juan Carlos Arroyo Portero
Ed.: Cinter Divulgación Técnica

APPENDIX 1. INTEGRAL'S TABLE

$\int_0^l M \cdot M' dx$						
	$M \cdot M' \cdot l$	$\frac{1}{2} M \cdot M' \cdot l$	$\frac{1}{2} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{2}{3} M \cdot M' \cdot l$	$\frac{1}{2} M \cdot M' \cdot l$
	$\frac{1}{2} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{6} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$
	$\frac{1}{2} M \cdot M' \cdot l$	$\frac{1}{6} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{12} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$
	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$	$\frac{1}{12} M \cdot M' \cdot l$	$\frac{1}{5} M \cdot M' \cdot l$	$\frac{1}{5} M \cdot M' \cdot l$	$\frac{7}{48} M \cdot M' \cdot l$
	$\frac{2}{3} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$	$\frac{1}{5} M \cdot M' \cdot l$	$\frac{8}{15} M \cdot M' \cdot l$	$\frac{5}{12} M \cdot M' \cdot l$
	$\frac{1}{2} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$	$\frac{1}{4} M \cdot M' \cdot l$	$\frac{7}{48} M \cdot M' \cdot l$	$\frac{5}{12} M \cdot M' \cdot l$	$\frac{1}{3} M \cdot M' \cdot l$

APPENDIX 2. FIXED END MOMENT'S TABLE

$$\text{Distribution Factor: } r = \frac{K}{\sum K_i}$$

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